Polynomials

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1 Roots of unity

1. (USAMO 1976) The polynomials $A(x), B(x), C(x), D(x)$ satisfy the equation

$$A(x^5) + xB(x^5) + x^2C(x^5) = (1 + x + x^2 + x^3 + x^4)D(x).$$

   Show that $A(1) = 0$.

2. A sequence $a_1, a_2, \ldots, a_n$ is called k-balanced if $a_1 + a_{k+1} + \cdots = a_2 + \cdots + a_{k+2} + \cdots = \cdots = a_k + a_{2k} + \cdots$. Suppose the sequence a_1, a_2, $\ldots$, a_{50} is k-balanced for $k = 3, 5, 7, 11, 13, 17$. Prove that all the values a_i are zero.

3. Let $P(x)$ be a monic polynomial with integer coefficients such that all its zeros lie on the unit circle. Show that all the zeros of $P(x)$ are roots of unity, i.e., $P(x)|(x^n - 1)^k$ for some $n, k \in \mathbb{N}$.

2 Integer divisibility

The main lesson, as illustrated by the first set of problems here, is that if $P(x)$ has integer coefficients, then $a - b \mid P(a) - P(b)$.

4. (a) (USAMO 1974) Let a, b, c be three distinct integers, and let P be a polynomial with integer coefficients. Show that in this case the conditions $P(a) = b$, $P(b) = c$, $P(c) = a$ cannot be satisfied simultaneously.

   (b) Let $P(x)$ be a polynomial with integer coefficients, and let n be an odd positive integer. Suppose that $x_1, x_2, \ldots, x_n$ is a sequence of integers such that $x_2 = P(x_1), x_3 = P(x_2), \ldots, x_n = P(x_{n-1})$, and $x_1 = P(x_n)$. Prove that all the x_i's are equal.[1]

   (c) (Putnam 2000) Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

   (d) (IMO 2006) Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial

$$Q(x) = \underbrace{P(P(\ldots(P(x)\ldots))}_{k\ P'\text{s}}$$

   Prove that there are at most n integers t such that $Q(t) = t$.

5. Let a, b, c be nonzero integers such that both $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $\frac{a}{c} + \frac{c}{b} + \frac{b}{a}$ are integers. Prove that $|a| = |b| = |c|$.

[1]This problem appeared in Reid Barton's MOP handout in 2005. Compare with the IMO 2006 problem.

6. (IMO Shortlist 2005) Let a, b, c, d, e and f be positive integers. Suppose that the sum $S = a+b+c+d+e+f$ divides both $abc+def$ and $ab+bc+ca-de-ef-fd$. Prove that S is composite.

3 Crossing the x-axis

For any continuous function (e.g. polynomial) f, if $f(a)$ and $f(b)$ have different signs for some $a < b$, then there must exist a $t \in (a, b)$ such that $f(t) = 0$.

7. (China 1995) Alice and Bob play a game with a polynomial of degree at least 4:

$$x^{2n} + \square x^{2n-1} + \square x^{2n-2} + \cdots + \square x + 1.$$

They fill in real numbers to empty boxes in turn. If the resulting polynomial has no real root, Alice wins; otherwise, Bob wins. If Alice goes first, who has a winning strategy?

8. (USAMO 2002) Prove that any monic polynomial of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

4 Lagrange and Chebyshev

Lagrange interpolation. If $(x_1, y_1), \ldots, (x_n, y_n)$ are points in the plane with distinct x-coordinates, then there exists a unique polynomial $P(x)$ of degree at most $n-1$ passing through these points, and it is given by the expression

$$P(x) = \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

You may have seen many problems that can be solved *directly* using interpolation (e.g., here are the values of $P(1), P(2), \ldots, P(n)$, what's the value of $P(n+1)$). The following problems require more subtle uses of interpolation.

Chebyshev polynomials. These are polynomials satisfying

$$T_n(\cos\theta) = \cos n\theta.$$

One can show using induction T_n is indeed a polynomial, and has integer coefficients, with leading coefficient 2^n. Chebyshev polynomials (including its variants) are often useful because they are nicely bounded in $[-1, 1]$, so that they often serve as equality cases. Specifically, we have

$$|T_n(x)| \leq 1, \text{ whenever } x \in [-1, 1].$$

Outside of $[-1, 1]$, the values of $T_n(x)$ can be found through $T_n\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right) = \frac{1}{2}\left(x^n + \frac{1}{x^n}\right)$ (why?).
A common variant of Chebyshev polynomials is the class of polynomials satisfying $P_n(2\cos\theta) = 2\cos n\theta$. One can show that $P_n(x)$ is a monic integer polynomial. It also satisfies $P_n(x+x^{-1}) = x^n + x^{-n}$.

9. Show that if $f(x)$ is a monic polynomial of degree $n-1$, and $a_1, a_2, \ldots, a_n$ distinct real numbers, then

$$\sum_{i=1}^{n} \frac{f(a_i)}{\prod_{j \neq i}(a_j - a_i)} = 1$$

10. (IMO Shortlist 1997) Let f be a polynomial with integer coefficients and let p be a prime such that $f(0) = 0$, $f(1) = 1$, and $f(k) \equiv 0$ or $1 \pmod{p}$ for all positive integers k. Show that $\deg f \geq p - 1$.

11. Let P be a polynomial of degree n with real coefficients such that $|f(x)| \leq 1$ for all $x \in [0, 1]$. Show that $|f(-\frac{1}{n})| \leq 2^{n+1} - 1$.

12. Let $P(x)$ be a monic degree n polynomial with real coefficients. Prove that there is some $t \in [-1, 1]$ such that $|P(t)| \geq \frac{1}{2^n}$.

13. (Walter Janous, Crux) Suppose that $a_0, a_1, \ldots, a_n$ are real numbers such that for all $x \in [-1, 1]$, $|a_0 + a_1 x + \cdots + a_n x^n| \leq 1$. Show that for all $x \in [-1, 1]$, $|a_n + a_{n-1} x + \cdots + a_0 x^n| \leq 2^{n-1}$

14. Let $x_1, x_2, \ldots, x_n$, $n \geq 2$, be n distinct real numbers in the interval $[-1, 1]$. Prove that
$$\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \geq 2^{n-2},$$
where $t_i = \prod_{j \neq i} |x_j - x_i|$.

5 Irreducibility through mods

In abstract algebra language, if A is a UFD (unique factorization domain), then so is $A[x]$. In particular, fields are automatically UFDs, so that $K[x]$ is a UFD whenever K is a field. Useful examples of UFDs include: $\mathbb{Z}[x]$, $\mathbb{R}[x]$, $\mathbb{C}[x]$, $\mathbb{Z}[x, y]$, $\mathbb{F}_p[x]$.

The last example is especially worth mentioning. Yes, unique factorization holds even when the coefficients of the polynomial is considered in mod p (where p must be prime). This means that when we are considering factorizations of integers polynomials $f(x) = g(x)h(x)$, it may be helpful to reduce the problem to $\bar{f}(x) = \bar{g}(x)\bar{h}(x)$, where the coefficients are considered in mod p.

15. (a) (Eisenstein's criterion) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients such that $p \mid a_i$ for $0 \leq i \leq n - 1$, $p \nmid a_n$ and $p^2 \nmid a_0$. Then $f(x)$ is irreducible.
    (b) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients such that $p \mid a_i$ for $0 \leq i \leq n - k$, $p \nmid a_k$ and $p^2 \nmid a_0$. Then $f(x)$ has an irreducible factor of degree greater than k.

16. Let p be a prime number. Prove that $x^{p-1} + x^{p-2} + \cdots + 1$ is irreducible. (This is the example that follows every exposition of Eisenstein's criterion.)

17. Let n be a positive integer. Prove that $(x^2 + x)^{2^n} + 1$ is irreducible.

6 Irreducibility through roots

Interestingly enough, when trying to prove that a certain integer polynomial is irreducible, it can be usual to examine its complex roots.

18. Let $f(x) = a_n x^n + a_{n-1} x^n + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients, such that $|a_0|$ is prime and
$$|a_0| > |a_1| + |a_2| + \cdots + |a_n|.$$
Show that $f(x)$ is irreducible.

19. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $0 < a_0 \le a_1 \le \cdots \le a_n$ are real numbers. Prove that any complex zero of the polynomial satisfies $|z| \le 1$.

20. Let p be a prime. Prove that $x^{p-1} + 2x^{p-2} + 3x^{p-3} + \cdots + (p-1)x + p$ is irreducible.

21. (Cohn's criterion) Suppose that $\overline{p_n p_{n-1} \cdots p_1 p_0}$ is the base-10 representation of a prime number p, with $0 \le p_i < 10$ for each i and $p_n \ne 0$. Show that the polynomial

$$f(x) = p_n x^n + p_{n-1} x^n + \cdots + p_1 x + p_0$$

is irreducible.

22. (Romania TST 2003) Let $f(x) \in \mathbb{Z}[x]$ be an irreducible monic polynomial with integer coefficients. Suppose that $|f(0)|$ is not a perfect square. Show that $f(x^2)$ is also irreducible.

7 Rouché's theorem (optional)

The following theorem from complex analysis can be useful in locating the zeros of a polynomial.

Theorem (Rouché)**.** Let f and g be analytic functions (e.g. polynomials) on and inside a simple closed curve $\mathcal{C}$ (e.g. a circle). Suppose that $|f(z)| > |g(z)|$ for all points z on $\mathcal{C}$. Then f and $f - g$ have the same number of zeros (counting multiplicities) interior to $\mathcal{C}$.

23. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with complex coefficients, and such that

$$|a_k| > |a_0| + |a_1| + \cdots + |a_{k-1}| + |a_{k+1}| + \cdots + |a_n|$$

for some $0 \le k \le n$. Show that exactly k zeros of P lie strictly inside the unit circle, and the other $n - k$ zeros of P lie strictly outside the unit circle.

24. (Perron's criterion) Let $P(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with $a_0 \ne 0$ and

$$|a_{n-1}| > 1 + |a_{n-2}| + \cdots + |a_1| + |a_0|.$$

Then $P(x)$ is irreducible.

25. (IMO 1993) Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

26. (Romania ??) Let $f \in \mathbb{C}[x]$ be a monic polynomial. Prove that we can find a $z \in \mathbb{C}$ such that $|z| = 1$ and $|f(z)| \ge 1$.