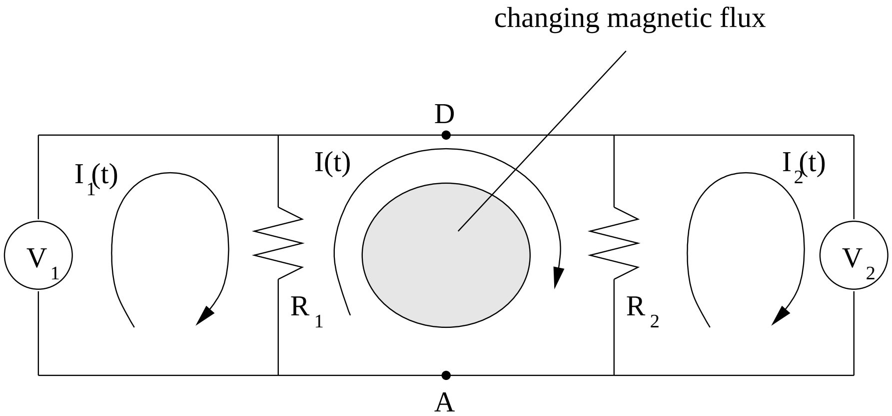
Massachusetts Institute of Technology - Physics Department

Physics 8.02 March 15, 2002.

Non-Conservative Fields - Do Not Trust Your Intuition!

The following notes may help you to digest the very *non-intuitive* consequences of Faraday's Law as discussed and demonstrated in my lecture of Friday, March 15.



A magnetic field is present in the shaded area; it is perpendicular to the page, and it is changing in time. There are two identical voltmeters V_1 and V_2, and two resistors R_1 and R_2; the internal resistance of each voltmeter, R_i, is much much larger than R_1 and R_2. All connecting wires have a negligible resistance.

We identify three **closed loops** in this circuit: the left loop with voltmeter V_1 and resistor R_1, the middle loop with the two resistors, and the right loop with voltmeter V_2 and resistor R_2. We call the currents in these loops $I_1(t)$, $I(t)$, and $I_2(t)$, respectively (see the diagram). We will assume that at time t the current in each loop is clockwise. If one (or more) of our currents turns out to be negative, it simply means that that current is counter clockwise.

In both the left and right closed loop we may apply Kirchhoff's 2nd rule:

$$\oint \vec{E} \cdot d\vec{l} = 0 \tag{1}$$

However, in the middle closed loop this rule can **NOT** be used; instead we now MUST use Faraday's Law which ALWAYS holds, we could also have used Faraday's Law for the left and right loops (the right side of Faraday's Law would then have been zero):

$$\oint \vec{E} \cdot d\vec{l} = \mathcal{E}(t) = -\frac{d\phi_B}{dt} \tag{2}$$

NOTICE: the induced EMF, $\mathcal{E}$, only depends on the magnetic flux change through any open surface that we attach to the closed loop, thus it depends on the change in the B field in the shaded area (since we may choose any open surface, I suggest we choose a flat surface that lies in the plane of our paper). $\mathcal{E}$ is therefore independent of R_1 and R_2.

Applying the above equations, starting at A, and going clockwise around each loop separately, we have:

Left loop (eq. 1)	$I_1R_i + I_1R_1 - IR_1 = 0$	(3)
Middle loop (eq. 2)	$IR_1 + IR_2 - I_1R_1 - I_2R_2 = \mathcal{E}$	(4)
Right loop (eq. 1)	$I_2R_2 - IR_2 + I_2R_i = 0$	(5)

$R_i \gg R_1$ and $R_i \gg R_2$, therefore $I_1 \ll I$ and $I_2 \ll I$, and these equations can be simplified:

$$I_1R_i - IR_1 \approx 0 \qquad (6)$$
$$I\,(R_1 + R_2) \approx \mathcal{E} \qquad (7)$$
$$I_2R_i - IR_2 \approx 0 \qquad (8).$$

Each voltmeter will show a reading which depends on the current through that meter (I_1 through the voltmeter on the left, and I_2 through the voltmeter on the right). The scales of the meters have been calibrated to show a voltage which is the product of the current through the meter and the internal resistance of that meter. Thus the left voltmeter will read $|V_1| = I_1R_i$, and the right voltmeter will read $|V_2| = I_2R_i$. If we substitute this in eqs. (6) and (8) we find:

$$|V_1| = I_1R_i \approx IR_1 \quad (9), \text{ and}$$
$$|V_2| = I_2R_i \approx IR_2 \quad (10).$$

If we connect the "+" side of both voltmeters with the A-side in the diagram (and the "-" side with the D-side), the recorded values of V_1 and V_2 at any point in time will be ***opposite*** in sign, as demonstrated in lectures. This is immediately obvious when you realize that a clockwise current I dictates a clockwise current I_1 as well as a clockwise current I_2 (this follows from eqs. 6 and 8; I, I_1 and I_2 always have the same sign). This means that when the current goes through the left voltmeter from its "+" side (A) to its "-" side (D), thus when it will indicate a "positive" voltage, that the current through the right voltmeter then goes from its "-" side (D) to its "+" side (A), and thus its voltage reading will be "negative".

If R_1 and R_2 are known, for given $\mathcal{E}$ (at a given point in time; see equation 2), the current I can be calculated from eq. (7), and the voltage readings follow from eqs. (9) and (10), but they will have an opposite sign. **Notice** that $|V_2 / V_1| \approx R_2 / R_1$ (independent of I).

Example

Suppose: $\mathcal{E} = 1$ Volt (at a given instant in time when the magnetic flux, coming out of the paper, is increasing), $R_1 = 100\ \Omega$, $R_2 = 900\ \Omega$, and $R_i = 10^7\ \Omega$, then, using eq. (7), we find $I \approx 1.0 \times 10^{-3}$ A (clockwise), and using eqs. (9) and (10), we find $|V_1| \approx \frac{1}{10}$ Volt, and $|V_2| \approx \frac{9}{10}$ Volt, **and the polarities of V_1 and V_2 are opposite**!

In case you are interested in I_1 and I_2, using eq. (9), we find $I_1 \approx 1.0 \times 10^{-8}$ A, and using eq. (10), we find $I_2 \approx 9.0 \times 10^{-8}$ A. Notice how small they both are in comparison with I. Both I_1 and I_2 are in clockwise direction.

If R_1 were 5 Ω, and R_2 were 45 Ω, (thus both 20 times smaller than above), I would be about 2.0×10^{-2} A (20 times higher than above), but the values for V_1 and V_2 would be the same as before!

Thus, for the given ratio $R_2/R_1 = 9$, at any time

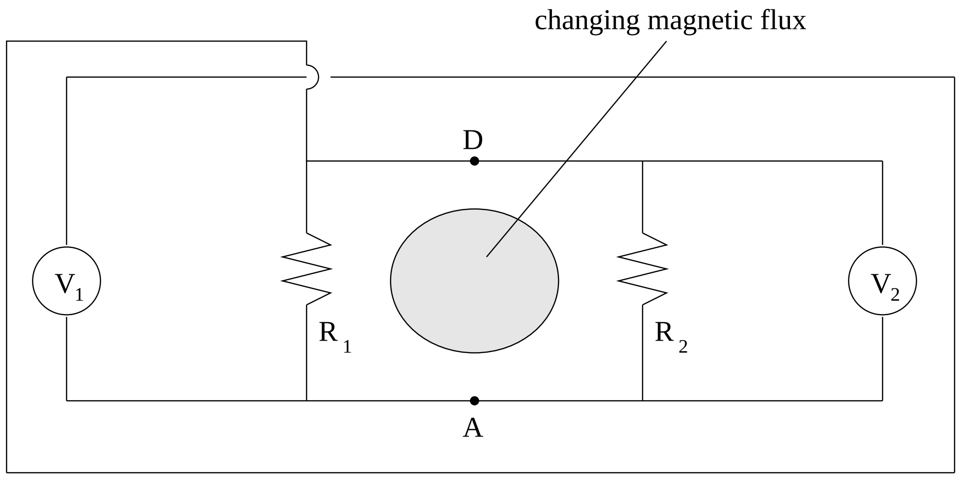
$|V_2| \approx \frac{9}{10}\mathcal{E}$ and $|V_1| \approx \frac{1}{10}\mathcal{E}$, **but the polarities are always opposite!**

Summary.
If I "travel" from A to D through the resistor R_1, the "potential difference"[1] between A and D is $\approx$ IR$_1$ (**A has a higher "potential"[1] than D**), and this value (V$_A$ - V$_D$) is registered by the left voltmeter. If I continue my journey through R_2, back to A, the "potential difference"[1] between D and A is $\approx$ IR$_2$ (**D has a higher "potential"[1] than A**), and this value (V$_D$ - V$_A$) is registered by the right voltmeter. Thus, once I have completed the closed loop journey in the middle loop, starting at A, and ending at A, the "potential difference"[1] V$_A$ - V$_A \neq 0$. Isn't that weird? **NO**, notice eq. (2)!

In Non-Conservative Fields, the electric potential difference[1], if one defines this as the integral of $\overrightarrow{E} \cdot d\overrightarrow{l}$ between two points, depends on the path, our intuition breaks down completely.

Test yourself.

Test 1. If you want to see whether you understand this difficult concept, calculate what the relative readings of V_1 and V_2 are for the diagram below. Notice that the wire that was connecting the left voltmeter with the D-side in our diagram on page 1, now is again connected with the D-side, but it is wrapped once around the whole circuit. The small arc in the wire above R$_1$ indicates that it is not in contact with the horizontal wire with which it appears to intersect due to the 2-dimensional projection. To stress this further I have interrupted the horizontal wire a trifle on either side of the arc. *This interruption is not real; the horizontal wire is continuous.*



Test 2. Now wrap that same wire not once around the whole circuit but 100 times before you connect it again on the D-side. Without realizing it, in doing so, you have been building yourself some kind of a transformer (we will discuss transformers later in the course). What now will the relative readings be between V_1 and V_2?

I hope this was helpful, this is not easy!

Walter Lewin

[1] *In non-conservative fields it may be better not to use the words "potential difference," but instead, one should say "the line integral of $\overrightarrow{E} \cdot d\overrightarrow{l}$ along a specified path."*