Adam S. Bolton
bolton@mit.edu

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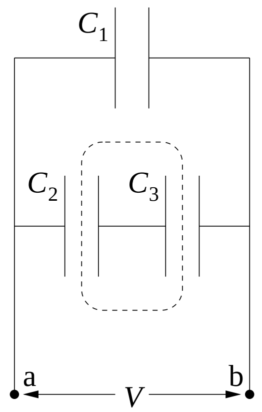
Assignment #3 Solutions

Problem 3.1

Capacitors in series and parallel. (Giancoli 24-23.)



(a) First we find the equivalent capacitance C_{23} of the C_2-C_3 combination. These two capacitors are combined in series, so we have

$$C_{23} = \left(\frac{1}{C_2} + \frac{1}{C_3}\right)^{-1} \quad .$$

C_{23} can then be combined in parallel with C_1, giving for the equivalent capacitance of the entire combination

$$C_{\rm eq} = C_1 + C_{23} = C_1 + \left(\frac{1}{C_2} + \frac{1}{C_3}\right)^{-1} \quad .$$

(b) In general we have $Q_1 = C_1 V_1$, $Q_2 = C_2 V_2$, and $Q_3 = C_3 V_3$. Given the imposed voltage V and the parallel wiring of C_1 with the C_2-C_3 combination, we must have $V_1 = V$ and $V_2 + V_3 = V$, or equivalently $Q_1/C_1 = V$ and $Q_2/C_2 + Q_3/C_3 = V$. In addition, charge conservation requires that $Q_2 = Q_3$ (consider the portion of the circuit enclosed by the dashed box in the figure above: charge cannot flow into or out of this part of the circuit, so if it carries no net charge initially, the charge on the right-hand plate of C_2 must always be equal and opposite to the charge on the left-hand plate of C_3). Letting $C_1 = C_2 = 2C_3 = C$, then, we find

$$Q_1 = CV, \quad Q_2 = Q_3 = CV/3 \quad .$$

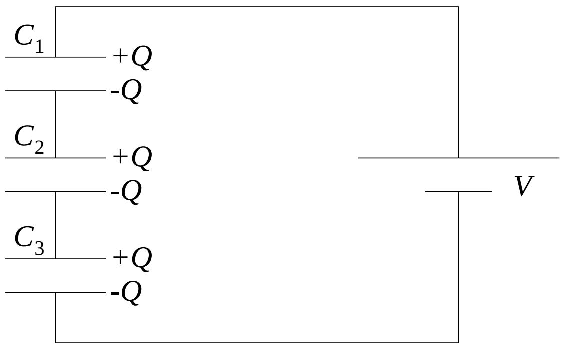
Plugging in the given numerical values,

$$Q_1 = 350\ \mu\text{C}, \quad Q_2 = Q_3 = 117\ \mu\text{C} \quad .$$

Problem 3.2

Switching capacitors.

(a) With switch B open and switch A closed, C_4 is out of the picture, and the circuit is effectively as shown in the diagram at right. If $V = 120$ V is the battery voltage, we must have

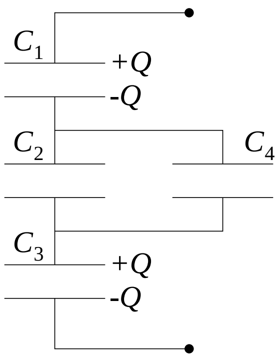


$$V_1 + V_2 + V_3 = V \quad .$$

By the same sort of charge-conservation argument as in problem 3.1, all three capacitors must necessarily have the same charge $\pm Q$ on their plates. Since all three have the same capacitance as well (call it "C"), we have

$$\begin{aligned} Q &= CV_1 = CV_2 = CV_3 \\ &\Rightarrow \; V_1 = V_2 = V_3 = V/3 = 40 \text{ V} \quad . \end{aligned}$$

Once the capacitors are charged in this manner, the potential across each will remain the same even after the opening of switch A.

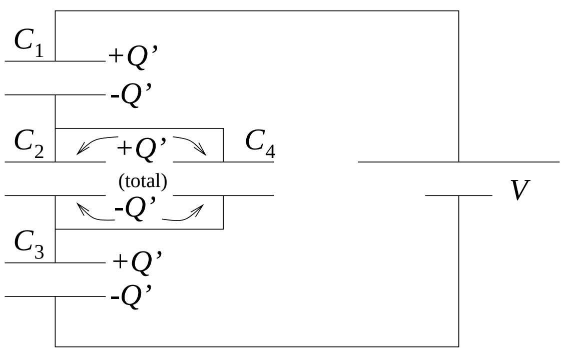


(b) Opening switch A has taken the circuit branch containing the battery out of the game, but not before charging capacitors C_1, C_2, and C_3 as described in part (a). Now, closing switch B brings C_4 back into play, and the circuit is effectively as shown at left. C_1 and C_3 will still each carry the same charge Q as before (opening switch A leaves them isolated from any flow of charge). However, the charge Q that originally resided on C_2 alone will now be distributed between C_2 and C_4. As $C_2 = C_4$, by symmetry we will have a charge $Q/2$ on each, and the voltage across C_2 and C_4 will thus be half the original voltage across C_2. So then,

$$V_1 = V_3 = 40 \text{ V} \;, \quad V_2 = V_4 = 20 \text{ V} \quad .$$

(c) With the capacitors initially uncharged and switch A open, closing switch B does not provide a potential across any of the capacitors, for the circuit branch containing the battery is effectively disconnected.

(d) Now we close switch A, with switch B previously closed. The circuit is as shown at right. The effective capacitance of the C_2-C_4 parallel combination is $C_{24} = C + C = 2C$. The entire four-capacitor system has an equivalent capacitance



$$\begin{aligned} C_{\text{eq}} &= \left(\frac{1}{C_1} + \frac{1}{C_{24}} + \frac{1}{C_3}\right)^{-1} \\ &= \left(\frac{1}{C} + \frac{1}{2C} + \frac{1}{C}\right)^{-1} = \frac{2}{5}C \quad , \end{aligned}$$

and thus the charge drawn from the battery and onto each of C_1, C_{24}, and C_3 is

$$Q' = C_{\text{eq}}V = \frac{2}{5}CV \quad .$$

Now we can determine the potential differences across the various capacitors:

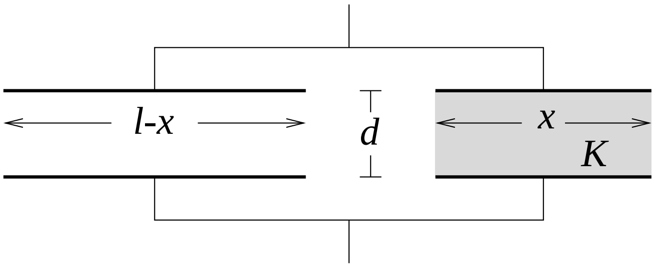
$$V_1 = V_3 = \frac{Q'}{C} = \frac{2}{5}V = 48 \text{ V}$$

and

$$V_2 = V_4 = \frac{Q'}{2C} = \frac{1}{5}V = 24 \text{ V} \quad .$$

Problem 3.3

The effect of a dielectric medium on the capacitance. (Giancoli 24-60.)



(a) To get the total capacitance of this arrangement, we consider it to be the equivalent of two capacitors in parallel, as shown above. The "capacitor" on the left (no dielectric) has plate area $l(l-x)$ and (by Giancoli equation 24-2, p. 615) capacitance $\epsilon_0 l(l-x)/d$. The "capacitor" on the right (with dielectric) has area lx and capacitance $K\epsilon_0 lx/d$. Thus the total capacitance is the sum of these:

$$C = \frac{\epsilon_0 l}{d}[(K-1)x + l] \quad .$$

When $x = 0$ this reduces to $\epsilon_0 l^2/d$, and when $x = l$ we get $K\epsilon_0 l^2/d$; these are the proper limits.

(b) With a potential difference V_0, the energy stored in this capacitor will be given by Giancoli equation 24-5 (p. 620):

$$U = \frac{1}{2}CV_0^2 = \frac{\epsilon_0 l}{2d}[(K-1)x + l]V_0^2$$

.

(c) Suppose we move the slab a little bit further in between the plates, increasing x by an amount Δx. The potential difference (provided by a battery, perhaps) remains constant while the capacitance increases a bit, so the change in stored energy will be

$$\begin{aligned} \Delta U &= \frac{1}{2}C_f V_0^2 - \frac{1}{2}C_i V_0^2 \\ &= \frac{\epsilon_0 l}{2d}(K-1)\Delta x V_0^2 \quad , \end{aligned}$$

which represents an increase (remember, $K > 1$). Does this mean that we have done positive work to move the slab in, pushing against an outward force? No it does not, for the battery has also done work on the capacitor/dielectric system! The charge on the capacitor has increased by an amount

$$\begin{aligned}\Delta Q &= C_f V_0 - C_i V_0 \\ &= \frac{\epsilon_0 l}{d}(K-1)\Delta x V_0 \quad ,\end{aligned}$$

and to put that charge there the battery has done an amount of work

$$W_{\text{battery}} = \Delta Q V_0 = \frac{\epsilon_0 l}{d}(K-1)\Delta x V_0^2 \quad .$$

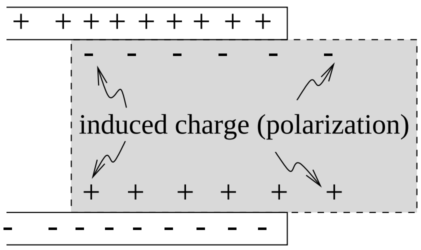
The work *we* have done is then related through

$$\begin{aligned}\Delta U &= W_{\text{battery}} + W_{\text{us}} \\ \Rightarrow W_{\text{us}} &= \Delta U - W_{\text{battery}} \\ &= -\frac{\epsilon_0 l}{2d}(K-1)\Delta x V_0^2 \quad .\end{aligned}$$

So we have actually done *negative* work on the system, and there must be an electric force pulling the slab inwards, balancing our force on the slab:

$$\begin{aligned}W_{\text{us}} &= F_{\text{us}}\Delta x = -F_{\text{elec}}\Delta x \\ \Rightarrow F_{\text{elec}} &= \frac{\epsilon_0 l}{2d}(K-1)V_0^2 \quad \text{(pulling slab inward.)}\end{aligned}$$

What is the nature of this electric force? The charge on the plates of the capacitor induces an opposite charge on the surfaces of the dielectric slab. At the edges of the capacitor, charge is induced on the slab even a bit beyond the capacitor plates (see diagram at right). This induced charge is attracted to the charge on the capacitor plates, and there is an inward component to this force, pulling the slab further in between the plates.



Problem 3.4

Comparing cylindrical and spherical capacitors.

(a) We compute the capacitance of the spherical capacitor using the result of Giancoli Example 24-3 (p. 617):

$$C_{\rm sph} = 4\pi\epsilon_0 \left(\frac{R_1 R_2}{R_2 - R_1}\right) = 4\pi(8.85\times10^{-12})\left(\frac{(0.06)(0.09)}{0.09-0.06}\right) = 2.00\times10^{-11}\ \mathrm{F} \quad .$$

For the cylindrical capacitor, we use the result of Giancoli Example 24-2 (p. 616):

$$C_{\rm cyl} = \frac{2\pi\epsilon_0 L}{\ln(R_2/R_1)} = \frac{2\pi(8.85\times10^{-12})(0.15)}{\ln(0.09/0.06)} = 2.06\times10^{-11}\ \mathrm{F} \quad .$$

These are nearly equal because the plate separation is the same in both cases, and the area of the plates is nearly the same, which (as we will show below) leads to about the same capacitance in the two cases.

(b) For $R_2 = R_1 + \delta$ ($\delta \ll R_1$), we have for the spherical capacitor

$$C_{\rm sph} = 4\pi\epsilon_0 \left(\frac{R_1(R_1+\delta)}{R_1+\delta-R_1}\right) = \epsilon_0 \frac{4\pi R_1^2}{\delta}(1+\delta/R_1) \quad .$$

As $\delta/R_1 \ll 1$, to leading order this is simply $C_{\rm sph} = \epsilon_0 A/\delta$, where $A = 4\pi R_1^2$ is the area of either shell. This is the parallel-plate capacitor formula.

For the cylindrical capacitor, we will make use of the following formula, which you may verify by means of a first-order Taylor expansion (see Giancoli Appendix A, p. A-1):

$$\text{for } x \ll 1, \quad \ln(1+x) \cong x \quad .$$

The capaitance is

$$C_{\rm cyl} = \frac{2\pi\epsilon_0 L}{\ln(1+\delta/R_1)} \cong \epsilon_0 \frac{2\pi R_1 L}{\delta} \quad .$$

Again, this is simply $C_{\rm cyl} = \epsilon_0 A/\delta$, where now $A = 2\pi R_1 L$ is the area of either of the cylindrical shells. So we see that whenever plate separation becomes very small compared to plate dimensions, even capacitors with curved geometry may be approximated as parallel-plate capacitors.

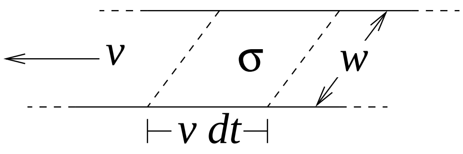
Problem 3.5

The Van de Graaff.

(a) We are given that the electric field on either side of the belt is $E = 10^6$ V/m. We can relate this to the surface charge density σ by the result of Giancoli Example 22-6 (p. 583):

$$E = \frac{\sigma}{2\epsilon_0} \quad \Rightarrow \quad \sigma = 2\epsilon_0 E = 2(8.85 \times 10^{-12})(10^6) = 18\ \mu\text{C/m}^2 \quad .$$

The belt is traveling with speed $v = 30$ m/s and has width $w = 0.5$ m. In a time interval dt a length $v\,dt$ of the belt passes any given position (see diagram at right). The area of the belt that passes this position will be $wv\,dt$, and the charge passing by in time dt will be $dQ = \sigma wv\,dt$. So the current carried by the belt is



$$I = \frac{dQ}{dt} = \sigma wv = (18 \times 10^{-6})(0.5)(30) = 2.7 \times 10^{-4}\ \text{A} \quad .$$

(b) The maximum electric field just outside the spherical dome will be the breakdown field for air, $E_{\text{max}} = 3 \times 10^6$ V/m (see Giancoli Example 23-5, p. 596, or problem 2.6 from Assignment #2.) The electric field just outside the surface of the dome will be related to the total charge on the dome by

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} \qquad \text{(Giancoli Example 23-4, p. 596),}$$

so the maximum charge that the dome can hold is

$$Q_{\text{max}} = 4\pi\epsilon_0 R^2 E_{\text{max}} = 4\pi(8.85 \times 10^{-12})(3 \times 10^6)R^2 = (3.3 \times 10^{-4}\ \text{C})R^2$$

(for R measured in meters). If the dome is being charged at a rate of $I = 2.7 \times 10^{-4}$ Coulombs/second, then the time required to charge the dome will be

$$\Delta t = Q_{\text{max}}/I = (1.2\ \text{s})R^2 \quad .$$

(c) The maximum electrostatic potential of the dome is (again, see Giancoli Example 23-5, p. 596)

$$V_{\text{max}} = RE_{\text{max}} = (3 \times 10^6\ \text{V})R \quad .$$

(d) For $R = 0.15$ m,

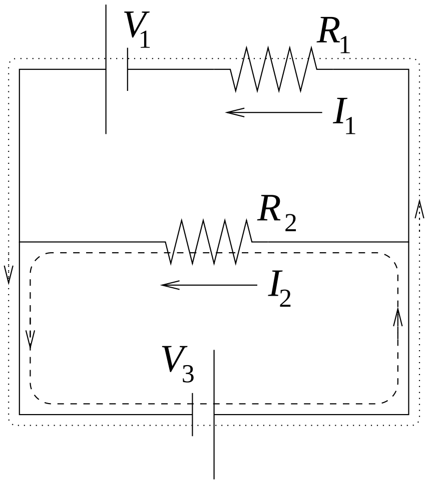
$$Q_{\text{max}} = 7.4\ \mu\text{C}, \quad \Delta t = 0.027\ \text{s} \quad , \quad \text{and} \quad V_{\text{max}} = 4.5 \times 10^5\ \text{V} \quad .$$

For $R = 0.5$ m,

$$Q_{\text{max}} = 83\ \mu\text{C}, \quad \Delta t = 0.3\ \text{s} \quad , \quad \text{and} \quad V_{\text{max}} = 1.5 \times 10^6\ \text{V} \quad .$$

Problem 3.6

Resistor circuit. (Giancoli 26-25.)



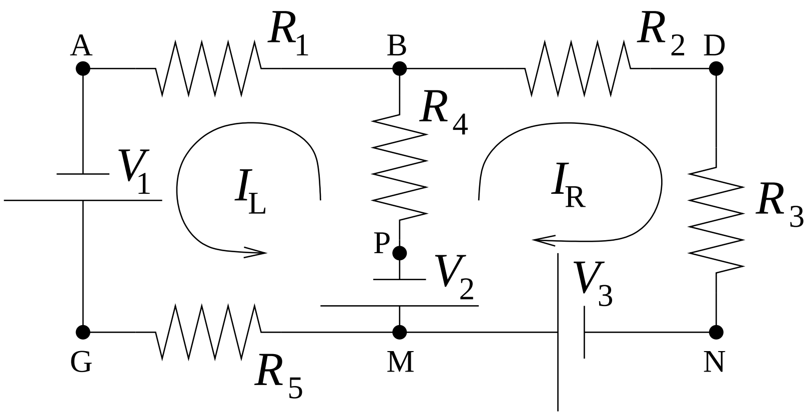
Let I_1 and I_2 be the currents in resistors R_1 and R_2 respectively, with the sense of positive current flow as shown in the diagram at right. Kirchhoff's loop rule applied to a loop around the outside of the circuit (dotted path in diagram) gives us the relation $V_1 + V_3 = I_1 R_1$. Taking a second loop around just the lower half of the circuit (dashed path in diagram) gives $V_3 = I_2 R_2$. Solving for the currents and plugging in given voltages and resistances, we have

$$\begin{aligned} I_1 &= (V_1 + V_3)/R_1 = 0.68 \text{ A} \quad , \\ I_2 &= V_3/R_2 = 0.4 \text{ A} \quad . \end{aligned}$$

Both I_1 and I_2 are positive, so the current through each resistor is directed "to the left".

Problem 3.7

Resistor network.



$$(R_1, R_2, R_3, R_4, R_5) = (10, 30, 50, 70, 100)\Omega$$

$$(V_1, V_2, V_3) = (12, 24, 36)\text{V}$$

(a) We'll designate the currents in the left and right parts of the circuit by I_L and I_R, and take them to be in the sense shown in the diagram above. The current through R_4 and V_2 is then $I_L + I_R$ upward, by conservation of charge (Kirchhoff's junction rule). If we move around the left circuit, Kirchhoff's loop rule says that the sum of the EMFs and voltage drops must be zero. This gives

$$+V_1 - I_L R_5 - V_2 - (I_L + I_R)R_4 - I_L R_1 = 0 \quad .$$

The sign of V_1 is positive here because we are moving from the negative terminal to the positive terminal of that battery. In the case of V_2, we are moving from the positive to the negative terminal, giving us a negative sign. Moving in a loop around the right-hand part of the circuit gives

$$+V_3 - V_2 - (I_L + I_R)R_4 - I_R R_2 - I_R R_3 = 0 \quad .$$

If we rearrange these two equations, we get

$$\begin{aligned}(R_1 + R_4 + R_5)I_L + (R_4)I_R &= V_1 - V_2 \quad , \\ (R_4)I_L + (R_2 + R_3 + R_4)I_R &= V_3 - V_2 \quad .\end{aligned}$$

To save ourselves an algebraic headache, we can insert the given numerical values for the resistances and battery voltages (in SI units) at this point:

$$\begin{aligned}180I_L + 70I_R &= -12 \quad , \\ 70I_L + 150I_R &= 12 \quad .\end{aligned}$$

The solution of this system gives

$$I_L = -119 \text{ mA} \ , \quad I_R = 136 \text{ mA} \ , \quad I_L + I_R = 16.3 \text{ mA} \quad .$$

Having obtained a negative value for I_L, we realize that the current in the left-hand side of the circuit in fact flows in a *clockwise* sense. To summarize the current picture:

- 119 mA flows clockwise through R_1 and R_5,
- 136 mA flows clockwise through R_2 and R_3, and
- 16.3 mA flows upwards through R_4.

(b)

$$V_A - V_P = -(16.3 \times 10^{-3})(70) + (119 \times 10^{-3})(10) = 0.049 \text{ V} \quad .$$

Note that the potential drops as we go through the 70 Ω resistor (R_4), because we go through with the current, but rises as we go through the 10 Ω resistor (R_1), because we go through against the current.

$$V_P - V_N = +36 \text{ V} - 24 \text{ V} = 12 \text{ V} \quad .$$

$$V_G - V_D = +(136 \times 10^{-3})(30) + (119 \times 10^{-3})(10) + 12 \text{ V} = 17.3 \text{ V} \quad .$$

Problem 3.8

Wire resistance. (Giancoli 25-52.)

(a) The resistance of the wire is simply

$$R = V/I = (22.0 \times 10^{-3})/(750 \times 10^{-3}) = 29.3 \text{ m}\Omega \quad .$$

(b) From equation 25-3 of Giancoli (p. 640), we can solve for the resistivity of the wire in terms of its resistance, length l, and cross-sectional area A:

$$\rho = RA/l = (29.3 \times 10^{-3})\pi(10^{-3})^2/(5.00) = 1.84 \times 10^{-8} \ \Omega \cdot \text{m}$$

(c) The current density is simply the current per unit cross-sectional area in the wire:

$$j = I/A = (750 \times 10^{-3})/(\pi \times (10^{-3})^2) = 2.39 \times 10^5 \ \text{A/m}^2 \quad .$$

(d) From Giancoli equation 25-17 (p. 649), we have for the electric field in the wire

$$E = \rho j = (1.84 \times 10^{-8})(2.39 \times 10^5) = 4.40 \times 10^{-3} \ \text{V/m} \quad .$$

(e) Giancoli equation 25-14 (p. 648) gives us the free-electron number density (we'll ignore ±-sign issues):

$$n = j/ev_\text{d} = (2.39 \times 10^5)/((1.6 \times 10^{-19})(1.7 \times 10^{-5})) = 8.8 \times 10^{28} \ \text{electrons/m}^3 \quad .$$

Problem 3.9

Energy consumption of heater, etc. (Giancoli 25-61.)

(a) The household's various daily energy usages are:

$$\begin{aligned} U_{\text{heater}} &= 1.8\ \text{kW} \times 3.0\ \text{h} = 5.4\ \text{kWh} \\ U_{\text{lights}} &= 4 \times 0.1\ \text{kW} \times 6.0\ \text{h} = 2.4\ \text{kWh} \\ U_{\text{stove}} &= 3.0\ \text{kW} \times 1.4\ \text{h} = 4.2\ \text{kWh} \\ U_{\text{misc}} &= 2.0\ \text{kWh} \end{aligned}$$

So the total energy used by the household in a day is

$$U_{\text{total}} = U_{\text{heater}} + U_{\text{lights}} + U_{\text{stove}} + U_{\text{misc}} = 14\ \text{kWh} \quad ,$$

and their monthly bill will be

$$30 \times 14\ \text{kWh} \times \$0.105/\text{kWh} = \$44.10 \quad .$$

(b) Let's convert the energy yield of coal from kcal/kg to kWh/kg (see the inside front cover of Giancoli):

$$7000\ \text{kcal/kg} \times \frac{1\ \text{kWh}}{860\ \text{kcal}} = 8.12\ \text{kWh/kg} \quad .$$

A 35%-efficiency power plant will get 0.35×8.12 kWh/kg $= 2.85$ kWh/kg in usable energy out of the coal. So the amount of coal to be burned yearly to meet the needs of the household is

$$365 \times 14\ \text{kWh} \times \frac{1\ \text{kg}}{2.85\ \text{kWh}} \simeq 1800\ \text{kg} \quad .$$

Problem 3.10

Electric car. (Giancoli 25-72.)

(a) Converting the car's traveling speed to SI units gives $v = 40$ km/h $= 11.1$ m/s. The car motor must supply a force that can balance the 240-N retarding force at the traveling speed, so the required motor power is

$$P = Fv = (240)(11.1) = 2664 \text{ W} \simeq 3.6 \text{ horsepower}$$

(from inside front cover of Giancoli: 1 horsepower = 746 W).

(b) The total on-board energy when the batteries are all fully charged is

$$U = 26 \times 12 \text{ V} \times 52 \text{ A} \cdot \text{h} = 16.2 \text{ kWh} = 5.84 \times 10^7 \text{ J} \quad .$$

If d is the distance the car travels on one charge, we will have $U = Fd$, giving

$$d = U/F = (5.84 \times 10^7)/(240) = 2.43 \times 10^5 \text{ m} = 243 \text{ km} \quad .$$

END