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Assignment #4 Solutions

Problem 4.1

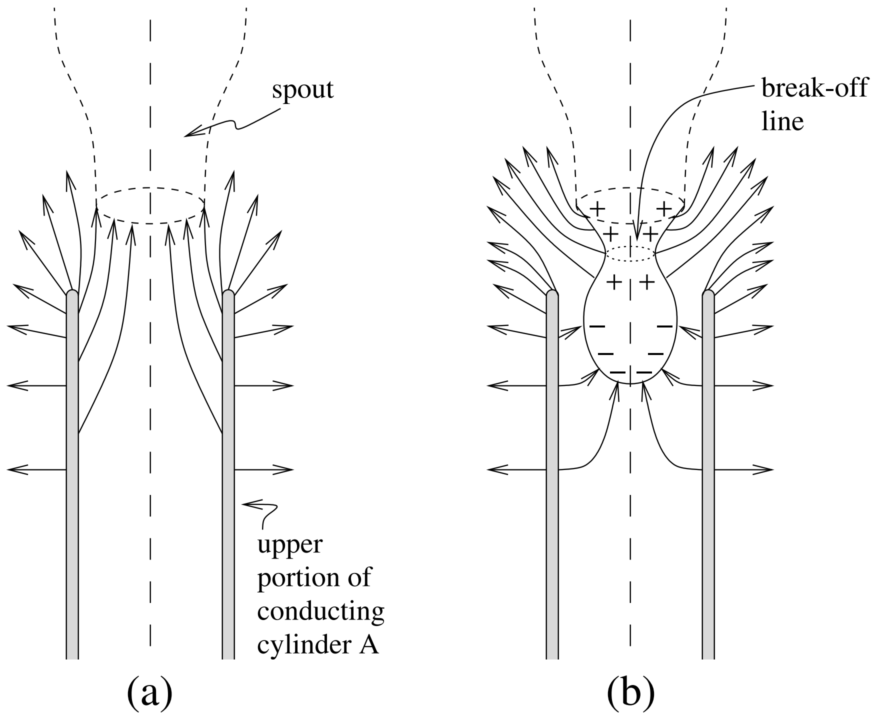
Kelvin Water Dropper.

Anyone who has not seen the "mysterious Water Dropper of Lord Kelvin" in operation is hard-pressed to comprehend how an initially "symmetric" device can develop "all by itself" such a strong asymmetry. After all, there seems to be no difference between left and right, yet one observes sparking, which implies a large potential difference between the water collectors on the left and on the right. After a brief reflection, however, one thing becomes clear: in order for the potential difference between the collectors to grow so quickly the water falling into the left cup must carry a net electric charge of the *opposite* sign relative to that falling into the right cup. Once we accept, as a working hypothesis, that the falling water on the left is, say, negatively charged *and* the falling water on the right is positively charged, we can see immediately how the buildup of the negative charge of the conducting system (B,D) and the positive charge of the conducting system (A,C) keeps growing until the critical value of $\sim 3 \times 10^6$ V/m field at the sparking site is reached. The remaining mysteries are where and how the water gets charged and why the tendency for the water to acquire a negative charge on one side of the apparatus leads to a positive charging-up of the water on the other side.

The site where there is most action is near the spout where the water comes out. To appreciate what is happening there, consider the following chronological sequence of hypothetical events near the upper rim of cylinder A:

1. By some tiny accidental perturbation, cylinder A acquires a minute charge. Let's say it's positive. If there were no water near the spout, the electric field lines at the upper rim would look something like the lines shown in sketch (a) below. Notice that they run predominantly upwards near the cylinder-spout axis.

2. In the water, there are OH^- and H^+ ions. If the water were pure and the PH were 7, 1 in 10^7 molecules would be dissociated into OH^- and H^+. As a result of the external field (as shown in (a)), the H^+ will be somewhat pushed up and the OH^- pulled down. Thus there is an induced charge separation similar to what we have seen demonstrated several times in lectures when a conductor is "exposed" to an external field. The charge distribution is now somewhat as shown in sketch (b). We only show one drop but the above charge separation works equally well for a continuous stream of water (as demonstrated in lecture).



3. Think of a stream of water as a stream of individual drops. It is manifestly clear from sketch (b) that, when the drop breaks off along the dotted line shown in the sketch, some of the upper (positive) polarization charge will be left behind in the water spout. Thus the separated water drop becomes *negatively charged*.

4. The next step is the automatic triggering of a perturbation of opposite sign on cylinder B. (The negative charge of the drop arriving in D distributes itself throughout the (B,D) equipotential surface). The perturbation field on site B has line patterns like those shown in sketch (a) except the directions of lines are *reversed*. The water drop that breaks off in cylinder B will carry a *positive* charge (opposite to the charge of the water drop formed inside cylinder A). The charge left behind in the right-hand side spout is, of course, negative.

This is the onset of an out-of-control process that feeds on itself: a minute perturbation grows larger and larger until a "disaster" strikes (here, the air breakdown). We witness here a "playful" example of a physical *instability*. (Our physical world is plagued with serious instabilities: violent weather (tornadoes, hurricanes), undesirable instabilities in the laboratory plasma that hamper our efforts to achieve controlled fusion, convective instability in the Sun's interior, violent flares at the solar surface, etc.)

It sometimes seems as if the growth of instability defies the law of energy conservation. This is of course not the case. Energy of *another* form is drained to feed the growing instability. In the "Kelvin Water Dropper" it is gravitational potential energy that feeds the machine.

Two last remarks to complete the story:

1. We started arbitrarily with a *positive* perturbation charge on A and ended up with a large positive charge on the (A,C) system. Assuming that the perturbation is initiated by chance with a *negative* trace charge changes nothing in the argument except that (A,C) is interchanged with (B,D). Since the initial perturbation is random in character, we cannot predict which column of falling water will turn out to be negative. Once A starts out negative, it remains that way as long as the water is running because the spark does not entirely discharge the system. However, the next day when we start up again, A could well be positive.

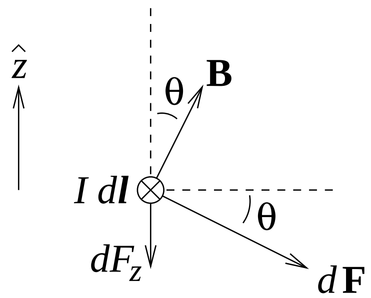
2. What about the fate of the charges left behind in the two spouts of the glass tubing? Notice that they are of opposite sign; thus the water in the tubing and the tank *does not* charge up. There is, however, an electric current in the water contained in the tubing. If both the tubing and the liquid were good insulators the machine would not work. The remaining charge at the spouts could not be removed (no current could flow through the liquid or tubing from one spout to the other), and the charge separation process in the liquid would quickly come to a halt (Why?). It is therefore essential that either the tubing or the liquid has modest conducting qualities.

Problem 4.2

Force on a current loop. (Giancoli 27-12.)

Let's take the z-direction to be along the axis of the loop of current, with positive z being in the overall direction of the magnetic field. The magnetic force on a differential segment of the wire (see figure) is given by Giancoli equation 27-4 (p. 691), and since the magnetic field is in fact perpendicular to the wire, we have

$$d\mathbf{F} = I\, d\boldsymbol{l} \times \mathbf{B} = I\, dlB \ , \quad \text{directed as shown in figure.}$$



From the azimuthal symmetry of the problem, we may conclude that the wire loop as a whole can only experience a force in the z-direction (any outward component of force on any particular segment of the wire will be exactly balanced by an oppositely-directed outward force component on an identical, diametrically opposite segment). So in calculating the total force on the loop, we need only add up (i.e., integrate) the z-components of the force on the many differential segments making up the loop. From our diagram, we see that this component for a typical segment has magnitude $|dF_z| = |d\mathbf{F}| \sin\theta$, and is directed towards negative z-values (downward). So the total force on the wire is

$$\mathbf{F} = \int d\mathbf{F} = \hat{z} \int dF_z = \hat{z} \int (-I\, dlB \sin\theta) = -2\pi r IB \sin\theta\, \hat{z} = \frac{-2\pi r^2 IB}{\sqrt{r^2+d^2}} \hat{z} \ .$$

(θ eliminated in favor of d using elementary trigonometry.)

Problem 4.3

Lorentz force on an electron. (Giancoli 27-20.)

The force experienced by a charged particle moving in a magnetic field is given by the Lorentz force law (Giancoli equation 27-5a, p. 692). Calculating the cross product explicitly in terms of Cartesian components, we have

$$\begin{aligned}
\mathbf{F} &= q\mathbf{v} \times \mathbf{B} \\
&= q[(v_y B_z - v_z B_y)\hat{x} + (v_z B_x - v_x B_z)\hat{y} + (v_x B_y - v_y B_x)\hat{z}] \\
&= q(v_x B_y - v_y B_x)\hat{z} \quad \text{(no other non-zero components)} \\
&= (-1.6 \times 10^{-19})[(4.0 \times 10^4)(0.60) - (-6.0 \times 10^4)(-0.80)]\hat{z} \\
&= (3.8 \times 10^{-15}\,\mathrm{N})\hat{z} \quad .
\end{aligned}$$

Problem 4.4

Spiraling electrons. (Giancoli 27-29.)

As indicated by Giancoli Conceptual Example 27-5 (p. 694), we think in terms of velocity components parallel and perpendicular to the magnetic field. Given the 45°-orientation of the velocity with respect to the field, we have

$$v_{\parallel} = v_{\perp} = v/\sqrt{2} \quad .$$

$v_{\perp}$ will give rise to the type of circular motion described in Giancoli Example 27-4 (p. 693), so we may calculate the radius of the loops using the results of that example:

$$r = \frac{mv_{\perp}}{qB} = \frac{mv}{\sqrt{2}qB} = \frac{(9.1 \times 10^{-31})(3.0 \times 10^6)}{\sqrt{2}(1.6 \times 10^{-19})(0.23)} = 5.2 \times 10^{-5}\,\mathrm{m} \quad .$$

The time it takes to complete one loop will be given by Giancoli equation (27-6) (p. 694):

$$T = \frac{2\pi m}{qB} \quad .$$

The electron moves along with constant speed $v_{\parallel}$ in the direction parallel to the magnetic field, so in a time T it will cover a distance between loops of

$$p = v_{\parallel}T = \frac{2\pi mv}{\sqrt{2}qB} = 2\pi r = 3.3 \times 10^{-4}\,\mathrm{m} \quad .$$

Problem 4.5

Torque on one winding of the rotor of your motor! (Giancoli 27-33.)

We'll use Giancoli equation (27-12) (p. 696) for the potential energy of a magnetic dipole in a uniform magnetic field:

$$U = -\mu B \cos\theta = -\boldsymbol{\mu} \cdot \mathbf{B}$$

(For a rectangular loop of N turns and dimensions $a \times b$ carrying current I, the magnetic moment is $\mu = NIab$.)

(a)

$$\begin{aligned} W &= U(\theta = 180°) - U(\theta = 0°) \\ &= -\mu B \cos(180°) + \mu B \cos(0°) \\ &= \mu B(1+1) \\ &= 2\mu B \quad . \end{aligned}$$

(b)

$$\begin{aligned} W &= U(\theta = -90°) - U(\theta = 90°) \\ &= -\mu B \cos(-90°) + \mu B \cos(90°) \\ &= \mu B(0+0) \\ &= 0 \quad . \end{aligned}$$

Problem 4.6

Mass spectrometer. (Giancoli 27-49.)

An ion of charge q accelerated from rest by a potential difference V will acquire kinetic energy $K = qV$. The particle then enters a B-field region, where it experiences a force perpendicular to its velocity and travels on a circular path with its speed and kinetic energy remaining constant. Newton's second law relates the magnetic force on the particle to its centripetal acceleration, giving $qvB = mv^2/R$. Combining this with with $K = mv^2/2$, we find that $K = q^2B^2R^2/2m$ for such "cyclotron orbits". Equating both expressions for K and solving for particle mass, we find that $m = qB^2R^2/2V$, as required.

Problem 4.7

Acceleration of deuterons in a cyclotron. (Giancoli 44-10.)

(*Before attempting this problem, you'll probably want to read the "Cyclotron" section starting on page 1116 of Giancoli to get a sense for the workings of the cyclotron.*)

(a) In problem 4.6, we derived the following expression for the kinetic energy of charged particles in cyclotron trajectories:

$$K = \frac{q^2 B^2 R^2}{2m} \quad .$$

The charge of an ionized deuteron is $+e$, the charge of its single proton. Its mass can be found (in atomic mass units) in Giancoli Appendix D:

$$m = (2.014)(1.66 \times 10^{-27}\,\text{kg}) = 3.34 \times 10^{-27}\,\text{kg} \quad .$$

(Strictly speaking, we must subtract off the mass of one electron, which is included in the tabulated deuterium mass, in order to account for ionization. This has no effect on the numerical value at our working level of accuracy, though.)

We wish the particle to have a kinetic energy of 10 MeV $= 1.6 \times 10^{-12}$ J at the exit radius of 1.0 m. So we may solve for the necessary magnetic field strength:

$$B = \frac{\sqrt{2mK}}{qR} = \frac{\sqrt{2(3.34 \times 10^{-27})(1.6 \times 10^{-12})}}{(1.6 \times 10^{-19})(1.0)} = 0.65\,\text{T} \quad .$$

(If you work out the speed corresponding to this desired kinetic energy using the familiar $K = mv^2/2$ formula, you will find that $v \simeq 0.1c$, where $c = 3.0 \times 10^8$ m/s is the speed of light. So if the particle's energy were much higher, we would need to resort to the relativistic relation between speed and energy given in lecture.)

(b) The alternating voltage frequency required will be the cyclotron frequency (Giancoli eq. (44-2), p. 1116)

$$f = \frac{qB}{2\pi m} = \frac{1}{2\pi R}\sqrt{\frac{2K}{m}} = \frac{1}{2\pi(1.0)}\sqrt{\frac{2(1.6 \times 10^{-12})}{3.34 \times 10^{-27}}} = 4.9\,\text{MHz} \quad .$$

(c) The deuteron has one unit of electronic charge, so with every passage through the 22-kV gap it gains 22 keV of kinetic energy. It passes through the gap two times per revolution. Given the known 10-MeV final energy of the particle, we can compute the total number of revolutions N:

$$N = \frac{10^7}{2 \times 2.2 \times 10^4} = 227 \text{ revolutions} \quad .$$

(d) Each revolution takes a time $T = 1/f$ (where f is the cyclotron frequency found in part (b)). So the total time from start to exit for a deuteron is

$$\Delta t = NT = N/f = 4.6 \times 10^{-5}\,\text{s}$$

(e) Let $\Delta K = 2 \times 22\,\text{keV} = 7.0 \times 10^{-15}\,\text{J}$ be the kinetic energy gained by the deuteron in one full revolution. At a time t after entering the cyclotron, the deuteron will have completed $t/T = tf$ revolutions. Since the particle makes many revolutions before exiting the cyclotron, we may approximate its kinetic energy as a smooth function of time, rather than a function that increases by abrupt steps corresponding to passages through the gaps. We then have the following kinetic energy relation, which yields the deuteron's speed as a function of time:

$$\frac{1}{2}mv^2 = \Delta K t f \Longrightarrow v(t) = \sqrt{2\Delta K t f/m} \quad .$$

To find the total "amount of ground covered" over the course of the cyclotron trip (call it Δs), we integrate the speed with respect to time over the trajectory:

$$\Delta s = \int_0^{\Delta t} v(t)\,dt = \sqrt{2\Delta K f/m}\int_0^{\Delta t} t^{1/2} dt = \frac{2}{3}\sqrt{\frac{2\Delta K f}{m}}(\Delta t)^{3/2} \quad .$$

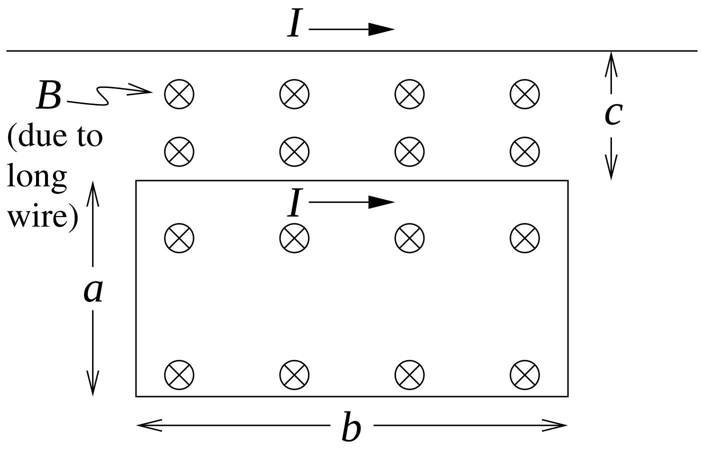
Eliminating Δt and f from this expression (and recognizing that $K/\Delta K = N$) leads to the charming result

$$\Delta s = \frac{4}{3}\pi N R \simeq 950\,\text{m} \quad .$$

Problem 4.8

Force on wire loop.
(Giancoli 28-14.)

(For convenience, let's adopt the symbols a, b, c, and I as shown in the diagram at right to represent the given physical quantities in this problem.)



The force on the current-carrying rectangular loop of wire will be due to the magnetic field of the long, straight wire. From Giancoli Sections 28-1 (p. 710) and 28-4 (p. 712), we know that this field will have magnitude

$$B = \frac{\mu_0 I}{2\pi r} \quad ,$$

where r is the perpendicular distance from the long wire. An application of the right-hand rule (see Giancoli Figure 27-9(b), p. 689) tells us that this magnetic field at the location of the rectangular loop is directed "into the page", as shown in the diagram above. We can now calculate the force on the rectangular loop by using

$$d\mathbf{F} = I\,d\boldsymbol{l} \times \mathbf{B} \qquad \text{(Giancoli Eq. (27-4), p. 691).}$$

For a small segment of the upper edge of the loop, $I\,d\boldsymbol{l}$ is directed to the right and $\mathbf{B}$ is into the page, so $d\mathbf{F}$ has magnitude $I\,dlB$, and by the right-hand rule it is directed upwards (that is, toward the long straight wire.) Since B has the constant value $\mu_0 I/2\pi c$ along the upper edge of the loop, the total force on that edge is simply

$$F_{\text{upper}} = \frac{\mu_0 I^2 b}{2\pi c} \quad , \quad \text{directed towards the long wire.}$$

Along the lower edge, a small segment has $I\,d\boldsymbol{l}$ directed to the left, with $\mathbf{B}$ still into the page, so now $d\mathbf{F}$ has magnitude $I\,dlB$ and is directed *downward* (i.e. *away* from the long wire). B has the value $\mu_0 I/2\pi(c+a)$ everywhere along the lower edge, so the force on this edge is

$$F_{\text{lower}} = \frac{\mu_0 I^2 b}{2\pi(c+a)} \quad , \quad \text{directed away from the long wire.}$$

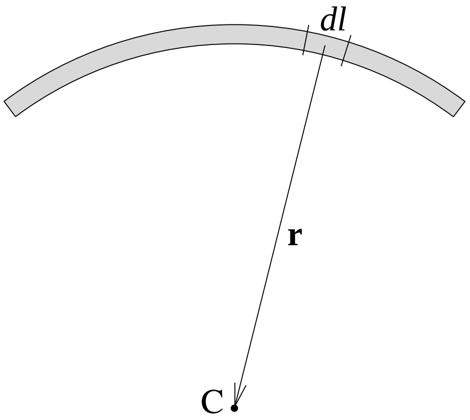
Now consider the left and right edges together. Specifically, consider two separate $I\,d\boldsymbol{l}$'s, one on the left edge and one on the right edge, with equal lengths and equal distances from the long wire. By virtue of the current circulation, the segment on the left edge will have an upward vector direction, while the one on the right edge will be directed downward. Both see the <u>same</u> B-field (magnitude and direction) from the long wire. So the $d\mathbf{F}$'s experienced by these paired $I\,d\boldsymbol{l}$'s will be of equal magnitude and opposite direction, and will contribute nothing to the <u>net</u> force on the wire loop. Thus upon integration the force on the left edge (which will be to the left) will be exactly cancelled by the force on the right edge (which will be to the right).

The net force on the loop is then

$$\begin{aligned} F &= F_{\text{upper}} - F_{\text{lower}} \\ &= \frac{\mu_0 I^2 b}{2\pi}\left(\frac{1}{c} - \frac{1}{a+c}\right) \\ &= \frac{(4\pi\times 10^{-7})(2.5)^2(0.1)}{2\pi}\left(\frac{1}{0.03} - \frac{1}{0.08}\right) \\ &= 2.6\times 10^{-6}\,\text{N} \ , \quad \text{directed towards the long wire.} \end{aligned}$$

Problem 4.9

Biot-Savart in action. (Giancoli 28-30.)



Consider first a small segment of length dl on one of the arcs (see figure at right). The Biot-Savart law (Giancoli equation (28-5), p. 719) gives the differential contribution of this segment to the field at C:

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{d\boldsymbol{l} \times \hat{r}}{r^2} \quad .$$

For both arcs, $\hat{r}$ is a unit vector in the direction of the displacement vector $\mathbf{r}$ from the segment to the point C, as shown. For the inner arc, $r = R_1$, and $I\, d\boldsymbol{l}$ is directed along the arc to the left, perpendicular to $\hat{r}$. So for segments along this arc, the field contribution at point C is

$$d\mathbf{B} = \frac{\mu_0 I\, dl}{4\pi R_1^2} \text{ “out of the page”.}$$

(Direction found using the right-hand rule for cross products.) This expression for $d\mathbf{B}$ does not depend upon the angular position of the tiny segment along the arc, so to get the total contribution of the inner arc to the magnetic field at C, we simply integrate dl to get the total arclength $R_1\theta$. Therefore,

$$\mathbf{B}_{\text{inner}} = \frac{\mu_0 I\, \theta}{4\pi R_1} \text{ “out of the page”.}$$

The analysis for the outer arc proceeds identically; $R_1 \to R_2$ and $I\, d\boldsymbol{l}$ is now directed to the *right* along the arc, flipping the direction of the field contribution:

$$\mathbf{B}_{\text{outer}} = \frac{\mu_0 I\, \theta}{4\pi R_2} \text{ “into the page”.}$$

Finally, consider any small segment along one of the straight sides: now $I\, d\boldsymbol{l}$ will be parallel to the displacement vector from the small segment to point C. So $I\, d\boldsymbol{l} \times \hat{r}$ will be zero, and there will be no contribution to the field at C coming from these straight sides.

The total magnetic field at C is thus given by (bearing in mind that $1/R_1 > 1/R_2$)

$$\mathbf{B} = \mathbf{B}_{\text{inner}} + \mathbf{B}_{\text{outer}} = \frac{\mu_0 I\, \theta}{4\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \text{ “out of the page”.}$$

END