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Assignment #7 Solutions

Problem 7.1

Ideal transformer. (Giancoli 29-42)

Let V_1 be the reading of a voltmeter attached across the primary (input) coil of the transformer and V_2 be the reading of a voltmeter attached across the secondary (output) coil. Also, let N_1, $\Phi_{B,1}$ and N_2, $\Phi_{B,2}$ be the number of turns and the magnetic flux per turn in the primary and secondary, respectively. Applying Faraday's law to a loop through the coil and voltmeter (for each coil) gives

$$\begin{aligned} V_1 &= N_1 \frac{d\Phi_{B,1}}{dt} \\ V_2 &= N_2 \frac{d\Phi_{B,2}}{dt} \quad . \end{aligned}$$

In an *ideal* transformer, $\Phi_{B,1} = \Phi_{B,2}$, and we have

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} \quad . \tag{1}$$

Another feature of an ideal transformer is that all the power delivered by an A.C. generator attached to the the primary is received by a load attached the secondary (no energy losses):

$$V_1 I_1 = V_2 I_2 \quad . \tag{2}$$

Note that (1) and (2) together give

$$N_1 I_1 = N_2 I_2 \quad .$$

In this problem, we are given

$$V_1 I_1 = V_2 I_2 = P = 100\,\mathrm{W}$$

and also

$$V_2 = 12\,\mathrm{V} \ , \quad I_2 = 26\,\mathrm{A} \quad .$$

This gives

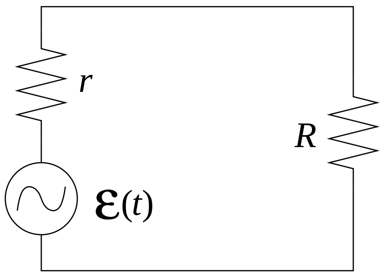
$$V_1 = P/I_1 = 100/26 \simeq 3.85\,\mathrm{W} \quad .$$

(a) Since $V_2 > V_1$, this is a step-up transformer.
(b) The voltage (again, defined as "the value read by a voltmeter attached across the coil") from primary to secondary is multiplied by a factor of

$$V_2/V_1 = V_2 I_1/P = (12)(26)/100 = 3.12 \quad .$$

Problem 7.2

A transformer for impedance matching.



(a) In this part, we have a simple circuit as shown at right, with $r = 0.4\,\Omega$, $R = 15\,\Omega$, $\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega t)$, $\mathcal{E}_0 = 150\,\mathrm{V}$, and $\omega/(2\pi) = 50\,\mathrm{Hz}$. From Ohm's law, the current $I(t)$ in the circuit is

$$I(t) = \frac{\mathcal{E}(t)}{r + R} \quad .$$

The instantaneous power delivered to the load is thus

$$P_R(t) = RI^2(t) = \frac{R\mathcal{E}_0^2}{(r+R)^2} \cos^2(\omega t) \quad ,$$

and the average power (by the definition of "average") is

$$\overline{P}_R = \frac{1}{T}\int_0^T P_R(t)\,dt = \frac{R\mathcal{E}_0^2}{(r+R)^2}\frac{1}{T}\int_0^T \cos^2(\omega t)\,dt$$

(T is the period of one full cycle: $T = 2\pi/\omega$).

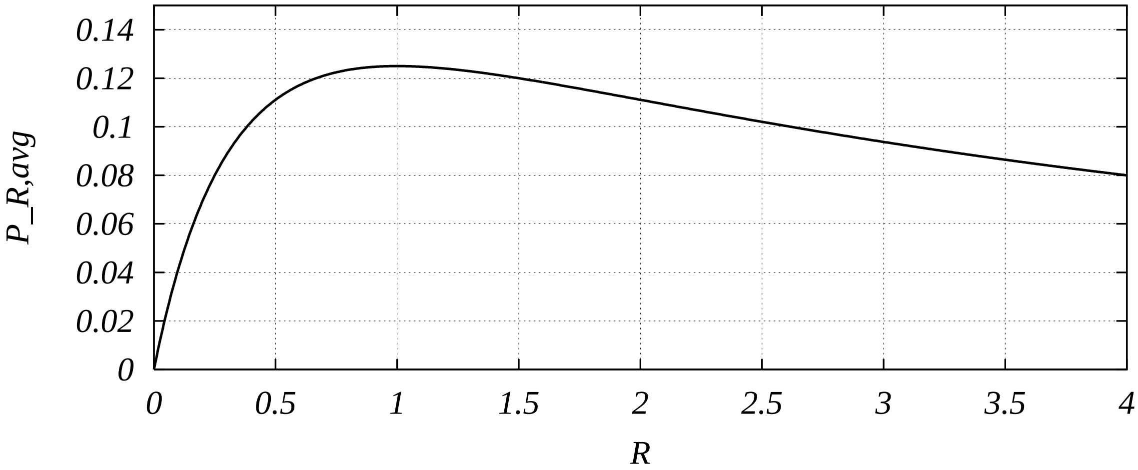
Since

$$\frac{1}{T}\int_0^T \cos^2(\omega t)\,dt = \overline{\cos^2(\omega t)} = \frac{1}{2} \quad ,$$

we have

$$\overline{P}_R = \frac{1}{2}\frac{R\mathcal{E}_0^2}{(r+R)^2} \quad .$$

If we consider $\overline{P}_R$ to be a function of R, we can easily show that it reaches a maximum when $R = r$. The following plot shows $\overline{P}_R$ (in units of $\mathcal{E}_0^2/r$) versus R (in units of r):

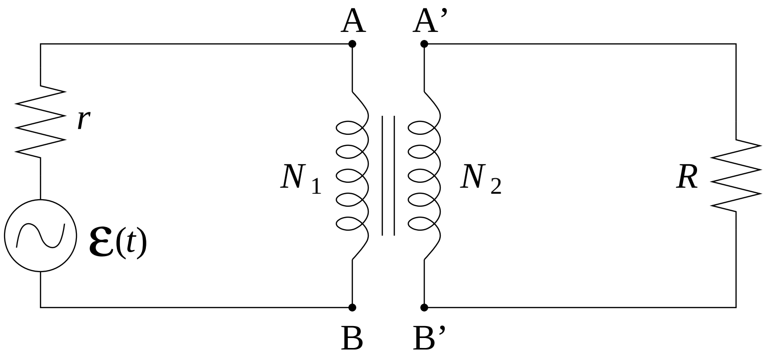


As stated in the problem, the maximum power is delivered to a load when the load "impedance" (R) is equal to the generator impedance (r). In the case under consideration we are *not* delivering maximum power to the load because $R \gg r$. Our given values for R, r, and $\mathcal{E}_0$ yield

$$\overline{P}_R = \frac{1}{2}\frac{(15)(150)^2}{(0.4+15)^2} \simeq 712\,\mathrm{W} \quad ,$$

which is far below the maximum possible power to the load.

(b) We now want to increase the power delivered to the load by matching impedances. We do this by inserting a transformer into the circuit as follows:



As in problem 1, we attach a voltmeter between points A and B across the primary transformer coil and call its reading V_1. Similarly, a voltmeter attached between A$'$ and B$'$ across the secondary reads a voltage that we will call V_2. For an ideal transformer and an A.C. voltage source, V_1 and V_2 defined in this manner will satisfy (1) and (2) from problem 1. For convenience, we'll define $\alpha \equiv N_1/N_2$ to be the ratio of turns between the two coils. The instantaneous power transferred to the load R is

$$P_2 = RI_2^2 \quad . \tag{3}$$

What is I_2? From the diagram,

$$V_1 = \mathcal{E} - rI_1 \quad ,$$

Making use of (1) and (2), this becomes

$$\alpha V_2 = \mathcal{E} - rI_2/\alpha \quad .$$

Since $V_2 = RI_2$ (Ohm's law), we have

$$\alpha(RI_2) = \mathcal{E} - rI_2/\alpha \quad ,$$

or solving for I_2,

$$I_2 = \frac{\mathcal{E}}{(\alpha R + r/\alpha)} \quad .$$

Plugging this into (3), we get

$$P_2 = \frac{R\mathcal{E}^2}{(\alpha R + r/\alpha)^2} = \frac{\alpha^2 R\mathcal{E}^2}{(r + \alpha^2 R)^2} \quad . \tag{4}$$

If we consider P_2 to be a function of α^2, it is small for both very small α^2 and very large α^2, so that it must go through a maximum at some intermediate value of α^2. To find that value, we differentiate (4) with respect to α^2 and set it to zero, giving

$$R\mathcal{E}^2 \left[\frac{1}{(r + \alpha^2 R)^2} - \frac{2\alpha^2 R}{(r + \alpha^2 R)^3} \right] = 0 \quad ,$$

or

$$1 - \frac{2\alpha^2 R}{r + \alpha^2 R} = 0 \quad .$$

Solving for α^2 gives

$$\alpha^2 = \left(\frac{N_1}{N_2}\right)^2 = \frac{r}{R} \Rightarrow \frac{N_1}{N_2} = \sqrt{\frac{r}{R}} = \sqrt{\frac{0.4}{15}} \simeq 0.163$$

for maximum power to load.

(c) The corresponding maximum value of P_2 is found from (4) with $\alpha^2 = r/R$:

$$P_2 = \frac{(r/R)R\mathcal{E}^2}{(r + (r/R)R)^2} = \frac{r\mathcal{E}^2}{4r^2} = \frac{\mathcal{E}^2}{4r} \quad .$$

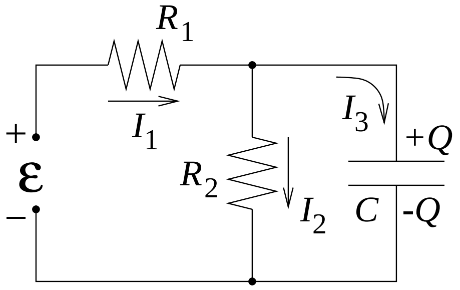
Time averaging gives

$$\overline{P}_2 = \frac{\mathcal{E}_0^2}{8r} = \frac{(150)^2}{8(0.4)} \simeq 7030\,\mathrm{W} \quad .$$

Compare this with our result from part (a).

Problem 7.3

RC circuit. (Giancoli 26-45.)



For convenience, define currents, charge, and polarity as shown in the diagram at right.

Two applications of Kirchhof's loop rule, one application of the junction rule, and the definition of current give us the following four equations:

$$\mathcal{E} = I_1R_1 + Q/C \tag{5}$$
$$\mathcal{E} = I_1R_1 + I_2R_2 \tag{6}$$
$$I_1 = I_2 + I_3 \tag{7}$$
$$I_3 = \frac{dQ}{dt} \ . \tag{8}$$

Subtracting (6) from (5), we get

$$\frac{Q}{C} - I_2R_2 = 0 \Rightarrow I_2 = \frac{Q}{R_2C} \ .$$

Substituting this into (7) to eliminate I_2 gives

$$I_1 = \frac{Q}{R_2C} + I_3 \ .$$

Now we plug this in for I_1 in (5):

$$\begin{aligned} \mathcal{E} &= \left(\frac{Q}{R_2C} + I_3\right) R_1 + \frac{Q}{C} \\ &= \left(1 + \frac{R_1}{R_2}\right) \frac{Q}{C} + R_1I_3 \ . \end{aligned}$$

Finally, making use of (8), we have

$$\mathcal{E} = R_1\frac{dQ}{dt} + \left(1 + \frac{R_1}{R_2}\right)\frac{Q}{C} \ .$$

This is now analogous to the differential equation for the charge on the capacitor in a simple *RC* circuit,

$$\mathcal{E} = R\frac{dQ}{dt} + \frac{1}{C}Q \ ,$$

with the replacements

$$R \to R_1 \ , \quad C \to \frac{C}{1 + R_1/R_2} \ . \tag{9}$$

Thus the problem is now solved by analogy to the simple *RC*-circuit case, as described in Giancoli section 26-4 (see pp. 669-671).

(a) The time constant "$\tau = RC$" for charging the capacitor is obtained using the replacements (9):

$$\tau = \frac{R_1 C}{1 + R_1/R_2} \quad .$$

(b) The charge on the capacitor as a function of time will be given by Giancoli Equation (26-5a) with the replacements (9):

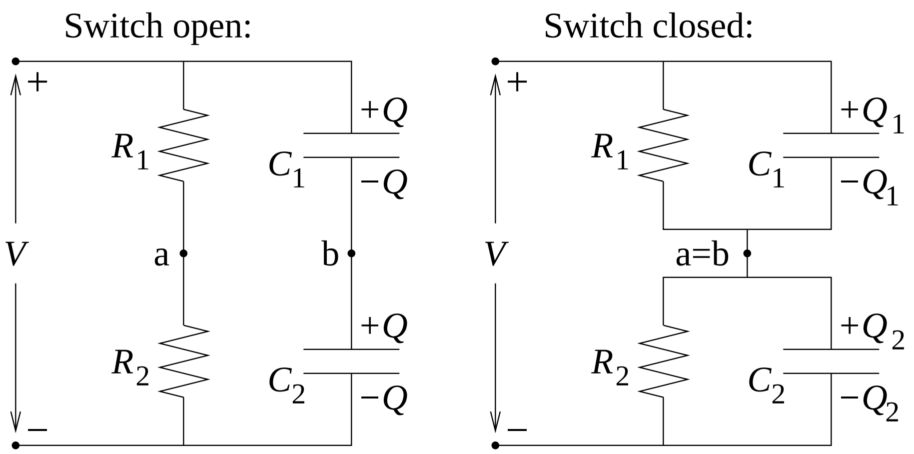
$$Q(t) = \frac{C\mathcal{E}}{1 + R_1/R_2}\left(1 - e^{-t/\tau}\right) \quad .$$

The maximum charge on the capacitor is then

$$Q_{\text{max}} = \lim_{t\to\infty} Q(t) = \frac{C\mathcal{E}}{1 + R_1/R_2} \quad .$$

Problem 7.4

RC circuit. (Giancoli 26-46.)



Define symbols as shown in the diagram, for algebraic convenience. Effective circuits for the open-switch and closed-switch cases are shown.

(a) With the switch open and the capacitors fully charged, there is no current in the capacitor branch of the circuit. For the resistor branch we simply have

$$V = (R_1 + R_2)I \quad \Longrightarrow \quad I = \frac{V}{R_1 + R_2} \quad .$$

Taking our zero-voltage to be at the negative terminal of the source, the potential at point "a" will simply be the voltage across R_2:

$$V_{\text{a}} = IR_2 = \frac{VR_2}{R_1 + R_2} = \frac{(24)(4.4)}{8.8 + 4.4} = 8\,\text{V} \quad .$$

(b) With the switch open, the two capacitors are wired in series, with an effective capacitance

$$C = \frac{C_1 C_2}{C_1 + C_2} \quad .$$

Each one will have the same charge $\pm Q$ on its plates, where

$$Q = CV = V\left(\frac{C_1 C_2}{C_1 + C_2}\right) \quad .$$

Relative to our chosen zero-point, the potential at "b" will be the voltage across C_2 alone:

$$V_{\mathrm{b}} = \frac{Q}{C_2} = \frac{VC_1}{C_1 + C_2} = \frac{(24)(0.48 \times 10^{-6})}{0.48 \times 10^{-6} + 0.24 \times 10^{-6}} = 16\,\mathrm{V} \quad .$$

(c) With the switch *closed* and the capacitors fully charged, the only current in the circuit is still a single current I passing entirely through both resistors, with

$$V = (R_1 + R_2)I \quad \Longrightarrow \quad I = \frac{V}{R_1 + R_2}$$

as before. The voltages across the individual resistors therefore remain the same. However, the voltages across the capacitors are now equal to the voltages across their corresponding resistors, unlike before. "a" and "b" have become effectively the same point in the circuit; the new potential at "b" is equal to the (unchanged) potential at "a", which is the voltage across R_2:

$$V_{\mathrm{b}}' = IR_2 = 8\,\mathrm{V} \quad \text{(same as part (a))}.$$

(d) Before the switch is closed, both capacitors carry a charge

$$Q = CV = V\left(\frac{C_1 C_2}{C_1 + C_2}\right) = \frac{(24)(0.48 \times 10^{-6})(0.24 \times 10^{-6})}{0.48 \times 10^{-6} + 0.24 \times 10^{-6}} = 3.84\,\mu\mathrm{C} \quad .$$

So the net charge on the bottom plate of C_1 and the top plate of C_2 (which together are isolated from charge flow from the rest of the circuit) is $-3.84\,\mu\mathrm{C} + 3.84\,\mu\mathrm{C} = 0$.

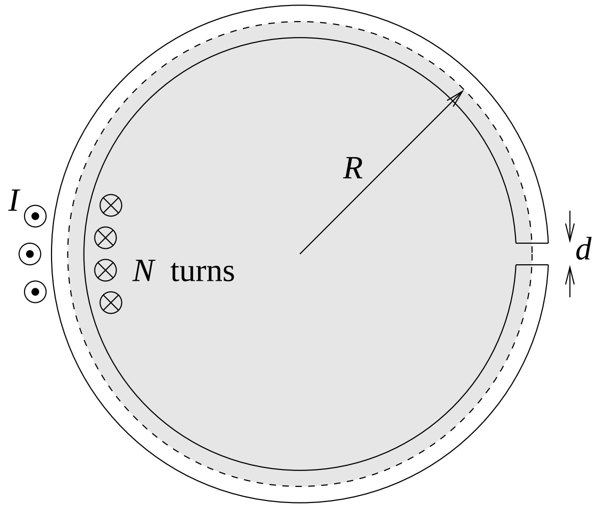
After the switch is closed, the voltage across C_2 is $V_2 = 8\,\mathrm{V}$, and the voltage across C_1 is $V_1 = 24 - 8 = 16\,\mathrm{V}$. The final charges on the two capacitors are therefore

$$\begin{aligned} Q_1 = C_1 V_1 = (0.48 \times 10^{-6})(16) &= 7.68\,\mu\mathrm{C} \quad , \\ Q_2 = C_2 V_2 = (0.24 \times 10^{-6})(8) &= 1.92\,\mu\mathrm{C} \quad . \end{aligned}$$

Now the net charge on the bottom plate of C_1 and the top plate of C_2 is $-7.68\,\mu\mathrm{C} + 1.92\,\mu\mathrm{C} = -5.76\,\mu\mathrm{C}$. So we may conclude that $\underline{-5.76\,\mu\mathrm{C}\text{ flows to the right}}$ through the switch after closing. (Or, +5.76 μC flows to the left, depending on your perspective!)

Problem 7.5

Electromagnet with small air gap.



We will assume that the magnetic field is azimuthal. Because there are no magnetic monopoles, the B-field lines form closed loops, and the number of lines will therefore be continuous across the air-steel interface in the gap. So the magnetic field strength in the gap will be approximately equal to the magnetic field strength in the steel.

In the presence of materials with non-zero magnetic susceptibility, Ampère's law (without Maxwell's correction) becomes

$$\oint \frac{1}{\kappa_M}\mathbf{B}\cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}} \quad ,$$

where κ_M is the relative permeability of the material (see Giancoli sections 28-9 and 28-10 (pp. 724-726), and take $K_{\text{m}} \leftrightarrow \kappa_M$). Applying this form of Ampère's law to the dashed loop and shaded open surface shown in the diagram (and defining I, N, R, and d as shown, for convenience):

$$\int_{\text{steel}} \frac{1}{\kappa_M}\mathbf{B}\cdot d\boldsymbol{l} + \int_{\text{gap}} \mathbf{B}\cdot d\boldsymbol{l} = \mu_0 N I$$

($\kappa_M = 1$ in air to a very good approximation: air is practically vacuum compared to steel). Note that the total current through our open surface is NI, because the wire cuts the surface N times. With our knowledge that B in the gap is (approximately) the same as B in the steel, we have

$$B\left[\frac{2\pi R - d}{\kappa_M} + d\right] = \mu_0 N I \quad ,$$

and since $2\pi R - d \simeq 2\pi R$,

$$B \simeq \frac{\mu_0 N I}{2\pi R/\kappa_M + d} = \frac{(4\pi\times 10^{-7})(120)(15)}{(2\pi)(0.07)/2500 + 2.5\times 10^{-3}} \simeq 0.85\,\text{T} \quad .$$

There are two points worth commenting on here. First, our assumption that B in the gap is the same as B in the steel is only good if the width of the gap is small compared to the cross-sectional radius of the electromagnet (2.5 mm compared to $\sim$ 1 cm here). In actual fact, fringing fields will cause the magnetic field in the gap to be less than in the steel, and the problem gets worse as the gap gets wider compared to the cross-sectional radius. Second, note that even though $d \ll 2\pi R$, the major contribution to the integral

$$\oint \frac{1}{\kappa_M} \mathbf{B} \cdot d\boldsymbol{l}$$

comes from the gap, because $\kappa_M = 1$ there. Over the *longer* path in the steel, the large value of κ_M there reduces the contribution to the line integral to *less* than that of the gap.

Problem 7.6

RC circuit.

(a) First consider the time interval $0 < t < T/2$. During this interval the driving voltage is constant ($V = V_0$), and we have

$$\begin{aligned} V_0 &= V_R + V_C \\ &= RI + Q/C \\ &= R\frac{dQ}{dt} + \frac{Q}{C} \quad . \end{aligned}$$

With $\tau = RC$, the general solution to this differential equation is

$$Q(t) = CV_0 + Ae^{-t/\tau} \quad ,$$

where A is an integration constant. To determine A, we consider our initial conditions. Since $V(t) = 0$ for $t < 0$, the charge Q on the capacitor is zero for $t < 0$ and also right at $t = 0$. For this to be true, we must have $A = -CV_0$, and thus

$$Q(t) = CV_0 \left(1 - e^{-t/\tau}\right) \quad \text{for} \quad 0 < t < T/2 \quad .$$

Since $C/\tau = 1/R$, we have for the current in the circuit

$$I(t) = \frac{dQ}{dt} = \frac{V_0}{R} e^{-t/\tau} \qquad (0 < t < T/2).$$

The voltage across the capacitor is Q/C, or

$$V_C(t) = V_0 \left(1 - e^{-t/\tau}\right) \qquad (0 < t < T/2).$$

The power delivered from the source (not to be confused with the power dissipated in the resistor!) is in general $P = IV(t)$, so for $V(t) = V_0$ during the first half-period,

$$P(t) = \frac{V_0^2}{R} e^{-t/\tau} \qquad (0 < t < T/2).$$

Now let's consider the interval $T/2 < t < T$. Here we have $V(t) = 0$, and the equation for Q becomes

$$R\frac{dQ}{dt} + \frac{Q}{C} = 0 \quad .$$

The general solution ($\tau = RC$) is now

$$Q(t) = Q_0 e^{-t/\tau} \qquad (T/2 < t < T),$$

where Q_0 is a constant to be determined. Since our solutions for $Q(t)$ in the intervals $T/2 < t < T$ and $0 < t < T/2$ must match at $t = T/2$, we must have

$$CV_0\left(1 - e^{-T/2\tau}\right) = Q_0 e^{-T/2\tau} \quad .$$

Now, $\tau = RC = (40)(150 \times 10^{-6}) = 6\,\mathrm{ms}$, while $T = 0.3\,\mathrm{s}$. So $e^{-T/2\tau} \ll 1$, and we may neglect that term relative to 1, giving

$$Q_0 = CV_0 e^{+T/2\tau}$$

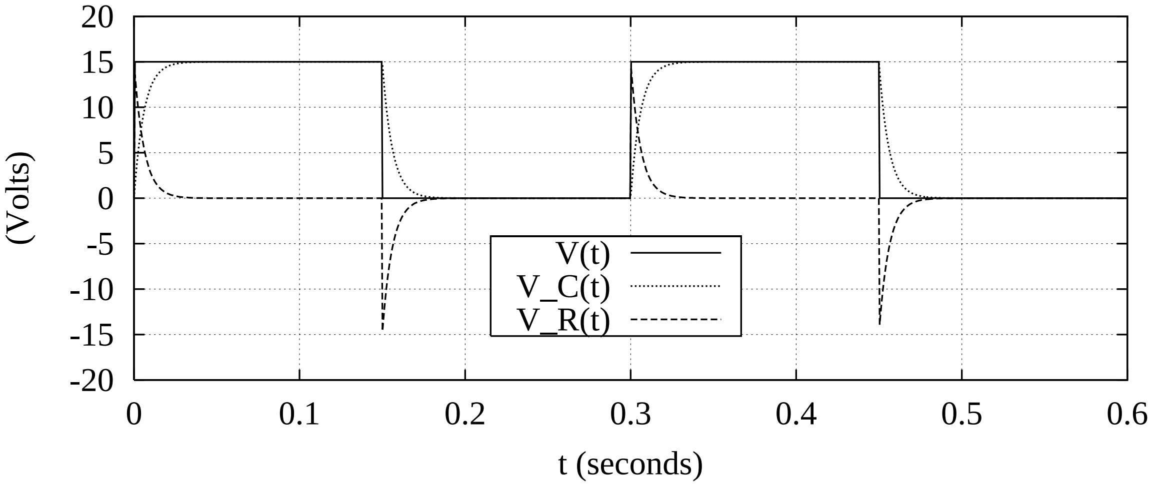
and thus

$$Q(t) = CV_0 e^{-(t-T/2)/\tau} \qquad (T/2 < t < T).$$

We then have

$$\begin{aligned} I(t) &= \frac{dQ}{dt} = -\frac{V_0}{R}e^{-(t-T/2)/\tau} \qquad (T/2 < t < T), \\ V_C(t) &= Q/C = V_0 e^{-(t-T/2)/\tau} \qquad (T/2 < t < T), \\ P(t) &= IV(t) = 0 \qquad\qquad (T/2 < t < T). \end{aligned}$$

(b) After one full period, the whole pattern repeats again. A graph of $V(t)$, $V_C(t)$, and $V_R(t) = IR$ over two full periods follows:



(c) The instantaneous power dissipated in the resistor is I^2R, so the energy dissipated in the resistor over one period is

$$\begin{aligned}
\int_0^T I^2 R\,dt &= 2\int_0^{T/2} I^2 R\,dt \\
&= 2\frac{V_0^2}{R}\int_0^{T/2} e^{-2t/\tau}\,dt \\
&= CV_0^2\left(1-e^{-T/\tau}\right) \\
&\simeq CV_0^2 \\
&= (150\times 10^{-6})(15)^2 \simeq 34\,\text{mJ} \quad .
\end{aligned}$$

The instantaneous power delivered by the power supply is $V(t)\,I$. The total energy delivered by the power supply over one period, is therefore

$$\begin{aligned}
\int_0^T V(t)\,I\,dt &= \int_0^{T/2} V_0 I\,dt \qquad (V(t) = 0 \text{ for } T/2 < t < T.) \\
&= \frac{V_0^2}{R}\int_0^{T/2} e^{-t/\tau}\,dt \\
&= CV_0^2\left(1-e^{-T/2\tau}\right) \\
&\simeq CV_0^2 \quad .
\end{aligned}$$

So, over one full cycle, the energy delivered by the power supply is equal to the energy dissipated as heat in the resistor. This result is inevitable from an energy-conservation point of view: over one full cycle, the capacitor is charged and then discharged, so no net energy is delivered to the capacitor, and *all* energy delivered by the power supply during one cycle *must* end up being dissipated in the resistor.

Energy Budget

We can now draw up a more detailed energy budget: towards the end of the first half of a cycle, the capacitor is essentially fully charged. It therefore contains $\frac{1}{2}QV_0 = \frac{1}{2}CV_0^2$ of electrostatic potential energy. During the subsequent discharge period (when $V(t) = 0$), this energy *must* come out in the resistor in the form of heat. Since the total energy dissipated during a full cycle is CV_0^2, we can conclude that an equal amount of energy $\frac{1}{2}CV_0^2$ was dissipated in the resistor during the charge-up period. This is obvious since the curve of current vs. time in one full cycle (T sec) shows two current "pulses" which are identical in shape. Their signs differ, but this difference disappears as the power dissipated in the resistor is proportional to I^2.

END