18.022 Notes Summary

January 12, 2021

The following summary has emerged from the Fall 2020 offering of 18.022, which was remotely taught by Professor Gigliola Staffilani. Most are from lectures and the course textbook (Colley's *Vector Calculus*), and some are derived from recitation material by Ms. Ruoxuan Yang.

The class is a more theoretical version of 18.02, dealing with topics in multivariable calculus (this shouldn't be a surprise given the title...) More specifically, the topics include vectors, functions, derivatives, arclength and parametrization, optimization and approximation, path and surface integrals, and finally ways to calculate these. Overall, the class was certainly more theoretical than 18.02. That being said, it remains a GIR and so not much proof-writing (if any) is expected from students.

As usual, these notes are by no means complete, and any error in them is on my part. They mostly contain definitions, theorems, and formulae that I found useful while reviewing. As such, they are not a good replacement for reviewing or practicing (in fact, nothing is). I have also emphasized and slightly exaggerated the theoretical part while ignoring the computational aspect out of personal interest. However, most of what was tested in the exams did not reach this level of abstraction.

1 Vectors

In a general mathematical sense, a vector is any element of a space that is equipped with *notions* of addition and scalar multiplication. However, for the purposes of this class, we focus on elements of $\mathbb{R}^n$.

Definition 1.1 — The dot product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, which we usually denote by $\mathbf{a} \cdot \mathbf{b}$ or $\langle \mathbf{a}, \mathbf{b} \rangle$, is given by:

$$a_1 b_1 + \cdots + a_n b_n$$

Formula 1.1.1 —

$$proj_{\mathbf{a}}\mathbf{b} = \mathbf{a} \cdot \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||^2}$$

Definition 1.2 — The cross product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, which we denote as

$$\mathbf{a} \times \mathbf{b},$$

is the unique vector $\mathbf{c}$ satisfying the following conditions:

1. $||\mathbf{c} = ||a|| \cdot |b|| \sin \theta||$,where θ is the angle between a and b

2. $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right-hand triplet.

Formula 1.2.1 —

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Theorem 1.3 — *Cauchy-Schwarz Inequality*

For any vectors v, w, we have

$$|\langle v, w \rangle|^2 \leq \|v\| \cdot \|w\|.$$

Equality occurs iff one of the vectors is a scalar multiple of the other.

This allows us to define the angle between two non-zero vectors in $\mathbb{R}^n$ to be the unique non-obtuse angle θ satisfying $\cos \theta = \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}$

2 Limits, Differentiability, and some Analysis

Definition 2.1 — ε, δ are usually used to denote teeeny-tiny positive numbers.

Definition 2.2 — Consider a function $f : \mathbb{R}^m \mapsto \mathbb{R}^n$. We say that $\mathbf{L}$ is the limit of f as $\mathbf{x}$ tends to $\mathbf{a}$ iff, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\mathbf{x}$ satisfying

$$||\mathbf{x} - \mathbf{a}||_{\mathbb{R}^m} < \delta,$$

we have that

$$||f(\mathbf{x}) - \mathbf{L}||_{\mathbb{R}^n} < \varepsilon$$

Definition 2.3 — The neighborhood of a point $\mathbf{x} \in X$ is a subset N that contains an open subset of X which in turn contains x.

Definition 2.4 — A set $X \subset \mathbb{R}^n$ is said to be **open** if every point in the X has a neighborhood that lies entirely within X. [2]

Definition 2.5 — A set $X \subset U$ is said to be **closed** in U if its complement $U \setminus X$ is open.

Remark — A set can be both closed and open. In this case, we sometimes call it **clopen**. Examples include the empty set $\emptyset$ and the whole number line when viewed as subsets of $\mathbb{R}$. Similarly, some sets are neither open nor closed.

Definition 2.6 — We say that $x \in X$ is an **accumulation point** of X if every neighborhood of x contains at least one element in X that is different from x. An alternative name often used in literature is **limit point**.

Theorem 2.7 — *A set is closed if and only if it contains all its accumulation points.*

Definition 2.8 — The directional derivative of $f : X \subset \mathbb{R}^n \mapsto \mathbb{R}$ along some unit vector $\mathbf{v}$ is given by

$$\partial_v f(\mathbf{x}) = \lim_{h \mapsto 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h},$$

provided the limit exists.

Definition 2.9 — The **partial derivative** of f with respect to a variable x_i, often denoted as f_{x_i} or $\frac{\partial f}{\partial x}$ is defined to be the directional derivative with of f along the basis vector $\mathbf{e}_i$.

Definition 2.10 — The **gradient** of f at a point $\mathbf{a}$ is the row matrix with entries being the partial derivatives of f evaluated at $\mathbf{a}$ in the correct order, provided these exist of course.

Definition 2.11 — We say that $f : X \subset \mathbb{R}^m \mapsto \mathbb{R}^n$ is differentiable at $\mathbf{a}$ if there exists a matrix $M \in \mathbf{R}^{n \times m}$ such that

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{||f(\mathbf{x}) - f(\mathbf{a}) - M \cdot (\mathbf{x} - \mathbf{a})||}{||\mathbf{x} - \mathbf{a}||} = 0.$$

Theorem 2.12 — *If f is differentiable at* $\mathbf{a}$*, i.e. if such an M exists, then $M = Df|_{\mathbf{a}}$, that is, the (ij) entry will be the partial derivative of the i-th component with respect to the j-th variable.*

Formula 2.12.1 — If $\partial_v f(\mathbf{x})$ exists and $f : X \subset \mathbb{R}^n \mapsto \mathbb{R}$ is differentiable, then

$$\partial_v f(\mathbf{a}) = \nabla f|_{\mathbf{a}} \cdot \mathbf{v}$$

Proof. Differentiability gives

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|f(\mathbf{x}) - f(\mathbf{a}) - \nabla f|_{\mathbf{a}} \cdot (\mathbf{x}-\mathbf{a})|}{||\mathbf{x}-\mathbf{a}||} = 0 \Longrightarrow \lim_{h\to 0} \frac{|f(\mathbf{a}+h\mathbf{v}) - f(\mathbf{a}) - \nabla f|_{\mathbf{a}} \cdot (h\mathbf{v})|}{|h|} = 0$$

$$\Longrightarrow \partial_v f(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{v}) - f(\mathbf{a})}{h} = \lim_{h\to 0} \frac{\nabla f|_{\mathbf{a}} \cdot (h\mathbf{v})}{h} = \nabla f|_{\mathbf{a}} \cdot \mathbf{v}$$

□

Theorem 2.13 — *Differentiability at* $\mathbf{a}$ *implies continuity at* $\mathbf{a}$.

Remark — The existence for partial derivatives at a point **DOES NOT** imply differentiability at that point. This is a common misconception, as single variable function are differentiable if the derivative exists. However, counter-examples exist to this statement in the case of multivariable functions. That being said, the following theorem may provide some consolation.

Theorem 2.14 — *Existence and continuity of all partial derivatives in some open neighborhood of* $\mathbf{a}$ *implies differentiability.*

Definition 2.15 — A function is said to be of class C^k if all its k-th partial derivatives exist and are continuous functions. A function in the classC^∞ is infinitely differentiable and continuous, and is often called **smooth**.

Remark — It can be proven that the above continuity classes each form a vector space over the field $\mathbb{R}$. Moreover, it is clear that if $i < j$ then $C^j \subset C^i$. After all, a function that is twice continuously differentiable is certainly continuously differentiable once. If we were to ask ourselves "what proportion of the functions in C^1 are also in C^2?" the answer would turn out, using Baire's Category Theory, to be "nearly none." If one had a way to randomly pick the a function in C^1, they will "almost never" get a function in C^2. This is analogous to the fact that when picking a random real number, we will "almost never" get a rational number (Dr. P.K).

Theorem 2.16 (Clairut's Theorem) — *If* $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *is of class* $C_,$ *then derivatives up to and including* k*-th partial derivatives commute. For example, if the function is* C^2*, then*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Theorem 2.17 (Implicit Function Theorem) — *Given* $\mathbf{F} : A \subseteq \mathbb{R}^{m+n} \mapsto \mathbb{R}^m$ *of class* C^1*, where* A *is open. Define*

$$S := \{(\mathbf{x}, \mathbf{y}) : \mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}\},$$

where $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^m$.

If the differential matrix of F *with respect to the* y *components is invertible when evaluated at* $(\mathbf{a}, \mathbf{b})$ *(i.e. non-zero determinant), then there exists*

1. *An open neighborhood* U *around* $\mathbf{a}$
2. *An open neighborhood* V *around* $\mathbf{b}$
3. *A* C^1 *function* $f : \mathbb{R}^n \mapsto \mathbb{R}^m$

such that

$$\mathbf{y} = f(\mathbf{x})$$

for all $\mathbf{x} \in U, \mathbf{y} in V$.

In other words, we can express a level surface of a function in two "multivariate" variables into one being a function in terms of the other.

3 Arclength and Some Differential Geometry

Definition 3.1 — We define a **partition** of an interval $[a, b]$ to be a finite set

$$\{a = t_0, t_1, \dots t_n = b\}.$$

Definition 3.2 — Suppose a partition $\mathcal{P}$ of $[a, b]$ is given by $\{a = t_0, t_1, \dots t_n = b\}$. We then define the length of this partition under $\mathbf{r} : [a, b] \mapsto \mathbb{R}^n$ to be:

$$l(\mathcal{P}) = \sum_{i=1}^{n} \|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\|$$

Definition 3.3 — A parametric curve $\mathbf{r} : [a, b] \mapsto \mathbb{R}^n$ is said to be **rectifiable** and of length L if the following are true:

1. $\mathbf{r}$ is continuous
2. For every partition $\mathcal{P}$ of $[a, b]$, $l(\mathcal{P}) \leq L$.
3. For every $\varepsilon > 0$, there exists a partition $\mathcal{P}$ such that $L \leq l(\mathcal{P}) + \varepsilon$

Theorem 3.4 — *If* $\mathbf{r} : [a, b] \mapsto \mathbb{R}^n$ *is differentiable, then:*

1. *The curve it defines is rectifiable.*
2. *The curve has length* $\int_a^b |\mathbf{r}'(t)| \, dt$.

Remark — Not all curves are rectifiable: a standard counter-example is the Koch snowflake. Given the above theorem, one may also wonder what happens if the function $r(t)$ is continuous but not necessarily differentiable. As it turns out, any rectifiable curve has a property known as *bounded variation* (BV), which therefore makes it "almost everywhere" differentiable (by a well-known theorem in Analysis).

Formula 3.4.1 — Arclength Parametrization for a differentiable curve $\mathbf{x}$.

$$s(t) = \int_a^t \|\mathbf{x}'(\tau)\| d\tau$$

Definition 3.5 — Unit Tangent

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Definition 3.6 — Curvature

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{\left\| \frac{d\mathbf{T}}{dt} \right\|}{\frac{ds}{dt}} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

Definition 3.7 — Normal

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{dt}}{\left\| \frac{d\mathbf{T}}{dt} \right\|}$$

Definition 3.8 — Binormal

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

Theorem 3.9 — *Frenet-Serret Formulae*

$$\frac{d}{ds}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}$$

4 Vector Fields

Definition 4.1 — A vector field $\mathbf{F}$ is a function $U \subset \mathbb{R}^n \longmapsto \mathbb{R}^n$.

Definition 4.2 — A vector field $\mathbf{F} : U \subset \mathbb{R}^n \longmapsto \mathbb{R}^n$ is said to be conservative if it is the gradient of some function, i.e. if there exists $f : U \subset \mathbb{R}^n \longmapsto \mathbb{R}$ such that

$$\mathbf{F} = \nabla f$$

In this case, one would call f the **potential** function of $\mathbf{F}$.

Theorem 4.3 — *If a C^1 vector-field* $\mathrm{F} = (F_1, \dots, F_n) : U \subseteq \mathbb{R}^n \longmapsto \mathbb{R}^n$ *is conservative, then*

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

over its domain. In other words, the derivative matrix of a conservative vector field over its domain is a symmetric square matrix.

Definition 4.4 — A **flowline** of a vector field $\mathbf{F} : U \subseteq \mathbb{R}^n \longmapsto \mathbb{R}^n$ is a differentiable parametrized curve $\mathbf{r} : [a, b] \longmapsto \mathbb{R}^n$ such that

$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{r}'(t)$$

over the domain, provided the expressions are well-defined.

Theorem 4.5 — *The potential of a conservative vector field is non-decreasing along itss flowline.*

Proof. Suppose $\mathbf{r}$ is the flowline and f is the potential. Then:

$$\frac{d}{dt}\big(f(\mathbf{r}(t))\big) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \|\mathbf{r}'(t)\|^2 \geq 0.$$

□

Corollary 4.5.1 — If $\mathbf{F}$ has a closed, non-constant flowline, then $\mathbf{F}$ is non-conservative.

Definition 4.6 (Divergence) —

$$div\mathbf{F} = \nabla \cdot \mathbf{F} := \frac{\partial F}{\partial x_1} + \dots \frac{\partial F}{\partial x_n}$$

Definition 4.7 (Curl) — For a differentiable vector field $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3)$, in *three-dimensional space*we define

$$curl\mathbf{F} = \nabla \times \mathbf{F} := \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Remark (Interpretations of Divergence) — If the divergence is positive at P, then P is a net source, meaning the net "fluid" leaving is greater than the net "fluid" entering, if we were to think of the vector field as depicting fluid motion. Similarly, negative divergence means that there is net influx of fluid, while zero divergence means that there is no net fluid motion (there could be motion, but it just enters and leaves at the same rate). This would be called an **incompressible** field.

Remark (Interpretations of Curl) — The curl represents the "angular rotation" of a twig dropped in the vector field of "fluid". The sign of the curl's components corresponds to whether the twig rotates counterclockwise (positive) or clockwise (negative). If the curl is the zero vector, than the "angular rotation" is zero and the vector field is often called **irrotational.**

Theorem 4.8 (Curl, Divergence, and Zeros) — *The following results hold:*

1. *The divergence of the curl of any vector field is zero.*
2. *The curl of any conservative field is the zero vector.*

5 Physical Applications of Integration

Formula 5.0.1 — Center of Mass of region W with density δ

$$\left(\frac{\iiint_W x\delta(x,y,z)dV}{\iiint_W \delta(x,y,z)dV}, \frac{\iiint_W y\delta(x,y,z)dV}{\iiint_W \delta(x,y,z)dV}, \frac{\iiint_W z\delta(x,y,z)dV}{\iiint_W \delta(x,y,z)dV}\right)$$

Formula 5.0.2 — Moment of Inertia about an Axis, where d is the distance to the axis.

$$\iiint_W d^2\delta(x,y,z)dV$$

Formula 5.0.3 — Mass of Wire under parametrization $\mathbf{r} : [a,b] \mapsto \mathbb{R}^3$ and density function $\delta(x,y,z)$.

$$\int_C \delta(x,y,z)ds = \int_a^b \delta(r(t)) \cdot \|\mathbf{r}'(t)\| \, dt$$

6 Line Integrals

Formula 6.0.1 — Line Integral of scalar field f over a path $\mathbf{r}$

$$\int_{\mathbf{r}} f\, ds = \int_a^b f(\mathbf{r}(t) \|\mathbf{r}'(t)\|\, dt$$

Formula 6.0.2 — Line Integral of vector field $\mathbf{F}$ over a path $\mathbf{r}$

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{r}(t) \cdot \mathbf{r}'(t)\, dt$$

Remark — A reparametrization of a scalar line integral yields the same result, whereas reparametrization the vector line integral may yield ± the answer, depending on the orientation.

Definition 6.1 — A vector field is said to be *path-independent* if the result of its line on any curve integral depends only on the starting and finishing points, not the path taken.

More on Vector Fields

Theorem 6.2 — *A continuous vector field is path-independent if and only if the vector field over any* ***closed****, simple, piecewise* C^1 *curve in its domain is* 0.

Definition 6.3 — A set $X \subset \mathbb{R}^n$ is said to be connected if any two points in X can be connected by a path whose image lines in X. This path **need not** be continuous.

Definition 6.4 — A set X is said to be path-connected if for any $A, B \in X$, there exists a continuous map $f : [0,1] \mapsto X$ such that $f(0) = A, f(1) = B$.

Theorem 6.5 — *Consider an arbitrary element P in a set X. The set X is path-connected if and only if for all $Q \in X$, there exists a continuous map as above with $f(0) = P, f(1) = Q$.*

Remark — The difference between the above two statements is subtle yet often useful. The first defines path-connectedness so that there exists such a map f between any two elements, while the second lets us fix our starting point and then just check if such a map exists from the starting point to any other element. That is, the second one is generally more practical for proving that a set is path-connected.

Definition 6.6 — A region of $\mathbb{R}^2$ or $\mathbb{R}^3$ is simply connected if every simple, closed curve in the set can be continuously shrunk to a point while remaining in the set.

Remark — Simple-connectedness implies path-connectedness, which in turn implies connectedness. Simple-connectedness is therefore the strongest condition. There are counterexamples to prove that the three notions are not equivalent (one of which is the infamous topologist's sine curve).

Theorem 6.7 — *Consider a C^1 vector field $\mathbf{F}$ whose domain is a simply connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$ for all points in its domain.*

7 Surfaces

Definition 7.1 — A parametrization of a surface in $\mathbb{R}^3$ is given by $\mathbf{r}(s,t) = \big(x(s,t), y(s,t), z(s,t)\big)$.

Definition 7.2 — Normal

$$\mathrm{N} := \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}$$

Definition 7.3 — A surface is smooth at a point (a, b) if both of the following are satisfied:

1. $\mathbf{r}$ is of class C^1 at some neighborhood of (a, b)
2. $\mathrm{N}(a, b) \neq \mathbf{0}$.

Formula 7.3.1 — Surface Area

Given a surface S which is defined by a quasi-one-one parametrization $\mathbf{r}$ of domain $D \subset \mathbb{R}^2$, we have:

$$\text{Surface Area of S} = \iint_D \|\mathrm{N}(s,t)\|\; ds\; dt = \iint_D \left\|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right\|\; ds\; dt$$

as

$$= \iint_D \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2}\; ds\; dt$$

Corollary 7.3.1 — The surface area of the graph of $f(x, y)$ over a domain D is given by

$$\iint_D \sqrt{f_x^2 + f_y^2 + 1}\; dx\; dy$$

Formula 7.3.2 — Surface Integral of Scalar Field

$$\iint_{\mathbf{r}} f\; dS = \iint_D f(\mathbf{r}(s,t)) \cdot \|\mathrm{N}(s,t)\|\; ds\; dt$$

Formula 7.3.3 — Surface Integral of Vector Fields over a domain D:

$$\iint_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{S} = .\iint_D \mathbf{F}(\mathbf{r}(s,t)) \cdot \mathrm{N}(s,t)\; ds\; dt$$

Remark — The same remark about reparametrization holds as the one for line integrals!

Definition 7.4 — A smooth, connected surface is said to be **orientable** if it possible to continuously assign each point a single unit normal vector.

Remark — A surface that is not orientable is said to be **non-orientable,** or one-sided. An example of this is the infamous Möbius strip.

Joke 7.4.1 — (From *The Big Bang Theory* Why did the chicken cross the Möbius Strip? To get the same side...

8 Fancy Theorems

Theorem 8.1 — *Green-Ostrogradsky Theorem*

Conditions:

1. *D is a closed, bounded region in* $\mathbb{R}^2$
2. *∂D is the union of finitely many simple, closed, piecewise C^1 curves.*
3. *We orient these curves so that the interior of D is to the left.*

Then:

$$\oint_{\partial D} M dx + N dy = \iint_D \left(N_x - M_y \right) dx\, dy$$

Theorem 8.2 — *Stokes' Theorem.*

Condition:

1. *S is a bounded, piecewise smooth, oriented surface in* $\mathbb{R}^3$.
2. *∂S is the union of finitely many simple, closed, piecewise C^1 curves.*

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_S \nabla \times \mathbf{F} \cdot n\, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

Theorem 8.3 — *Gauss' Theorem (a.k.a. Divergence Theorem)*

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \oiint_{\partial D} \mathbf{F} \cdot n\, dS = \oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S}$$

References

[1] Colley, Susan Jane *Vector Calculus , Fourth Edition.* Pearson, 2012.

[2] Sutherland, Wilson A. *Introduction to Metric Topological Spaces, Second Edition.* Oxford Mathematics, 2009.