Topology Notes

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The following notes evolved from my personal reading of Wilson Sutherland's "Introduction to Metric Topological Spaces." The book is a very well-written text that is quite easy to follow for anyone with prior exposure to mathematics. In all honesty, proof-writing skills and a good grasp of 18.01 material are the only prerequisites for a good deal of the book. As usual, these notes are likely to contain error, and are certainly incomplete. They are not a good replacement for practicing and doing mathematics yourself (in fact, nothing is). Instead, they were mostly meant as a crash-course in Topology that I needed for a UROP under the supervision of Professor Bjorn Poonen. Some of this content I had come across in 18.022 and 18.701, which spurred my curiosity on Topology. I should also credit Paige D. for pointing me to Andrew Lin's wonderful LaTeX package.

Definition 1 (Topological Spaces)
A topological space is a pair $(X, \mathcal{T})$ such that X is a set, and $\mathcal{T}$ is a family of subsets of X satisfying the following axioms:

1. $\emptyset, X \in \mathcal{T}$
2. The union of any collection of sets that lie in $\mathcal{T}$ must also lie in $\mathcal{T}$
3. The intersection of a **FINITE** number of sets that lie in $\mathcal{T}$ must also lie in $\mathcal{T}$

In this case, $\mathcal{T}$ is called a **topology** of X, and its elements are the open subsets of X. For brevity, we will often refer to the space as X when the topology is clear or implied.

Lemma 2
U is open in X if and only if for every element in U, there exists an open set U_x such that $x \in U_x \subseteq U$.

Definition 3
A map $f : (X, \mathcal{T}_x) \mapsto (Y, \mathcal{T}_y)$ is said to be continuous if for any U in $\mathcal{T}_y$, we have that $f^{-1}(U) \in \mathcal{T}_x$. Or as commonly told, f is continuous if the "pre-image of any open set is open."

Definition 4
A basis of a topological space $(X, \mathcal{T})$ is a subfamily $\mathcal{B} \subseteq \mathcal{T}$ such that every set in T is a union of sets in $\mathcal{B}$.

Proposition 5
If the pre-image of all basis elements is open under f, then the map is continuous.

Definition 6

V is said to be closed if $X \setminus V$ is open.

Example 7 (What continuity is (**not**))

It should be made clear that all continuity talks about is that the pre-images of open sets must be open. As we will prove later on, this is equivalent to the pre-images of closed sets being closed, and a myriad of other equivalences that allow us to express continuity in terms of multiple topological ideas. It does not say anything about the images of open sets, nor does it tell us what the images of closed sets are.

- When $f(x) = x^2$, $f(0,1) = (0,1)$: an open set is mapped to an open set.
- When $f(x) = x^2$, $f(-1,1) = [0,1)$: an open set is mapped to a set their is neither open nor closed.
- When $f(x) = e^x$, $f(-\infty, 0] = (0,1]$: a closed set is mapped to a set their is neither open nor closed.

Definition 8

A point $a \in X$ is said to be a point of closure of A in X if $U \cap A \neq \emptyset$ for all open subsets U containing a. The closure of A, denoted by $\overline{A}$, is the set of all closure points.

Theorem 9 (Properties of Closure)

The following properties hold for $A \subset X$:

1. $A \subseteq \overline{A}$.
2. $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.
3. A closed in X if and only if $\overline{A} = A$.
4. $\overline{A}$ is closed in X.
5. $\overline{\overline{A}} = \overline{A}$.
6. $\overline{A}$ is the smallest closed subset containing A.

Proposition 10 (Closure under Unions and Intersections)

$$\overline{\bigcup_{i=1}^{m} A_i} = \bigcup_{i=1}^{m} \overline{A_i}$$

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$$

Definition 11

A point a is said to be an interior point of A if there exists some open set containing a that lies in A. The set of all interior points is called the interior of A, and is denoted by $\mathring{A}$.

Theorem 12 (Properties of Interior)

The following properties hold for $A \subset X$:

1. $\mathring{A} \subseteq A$.
2. $A \subseteq B \Rightarrow \mathring{A} \subseteq \mathring{B}$.
3. A open in X if and only if $\mathring{A} = A$.
4. $\mathring{A}$ is open in X.
5. $\mathring{\mathring{A}} = \mathring{A}$.
6. $\mathring{A}$ is the largest open subset lying in A.

Proposition 13

$$\overline{X \setminus A} = X \setminus \mathring{A}$$

Proof. Take $t \in \overline{X \setminus A}$. As such, every open set containing t intersects $X \setminus A$, i.e. it contains an element outside of A. As such, $t \in X \setminus \mathring{A}$, as otherwise we would have an open set lying entirely in A, which is a contradiciton. This proves $\overline{X \setminus A} \subseteq X \setminus \mathring{A}$.

Now take $t \in X \setminus \mathring{A}$. That is, $t \notin \mathring{A}$, which means that no open set around t lies entirely within A, i.e. every open set around t intersects $X \setminus A$, and hence $t \in \overline{X \setminus A}$. Thus, $X \setminus \mathring{A} \subseteq \overline{X \setminus A}$.

□

Definition 14

The boundary of A, denoted by ∂A, is defined by:

$$\partial A := \overline{A} \setminus \mathring{A}$$

Lemma 15

The boundary of A in X also satisfies the following properties:

- $\partial A = \overline{A} \cap \overline{X \setminus A}$
- $\partial A = \partial(X \setminus A)$
- A is closed in X if and only if $\partial A \subseteq A$.
- $\partial A = \emptyset$ if and only if A is clopen in X.

Proof. We prove each one independently.

1. We establish that $\overline{A} \setminus \mathring{A} = \overline{A} \cap \overline{X \setminus A}$. If $t \in \overline{A} \setminus \mathring{A}$, then certainly $t \in \overline{A}$. Moreover since $t \notin \mathring{A}$, we conclude that no open set containing t lies entirely within A. This implies that $t \in \overline{X \setminus A}$. Similarly if $t \in \overline{A} \cap \overline{X \setminus A}$, then any open neighborhood of t intersects A but also non-emptily intersects the complement $X \setminus A$. That is, $t \notin \mathring{A}$ and hence $t \in \overline{A} \setminus \mathring{A}$, proving that $\overline{X \setminus A} \subseteq \overline{A} \setminus \mathring{A}$.

2. Try it on your own ;)
3. A is closed iff $\overline{A} \subseteq A$, which is equivalent to claiming that $\overline{A} \setminus \mathring{A} \subseteq A$.
4. $\partial A = \emptyset \Leftrightarrow \overline{A} \setminus \mathring{A} = \emptyset \Leftrightarrow \overline{A} = A = \mathring{A}$, which imply clopen-ness by 9 and 12.

□

Theorem 16

For $A \subseteq X$, the boundary ∂A is closed in X.

Proof. Using the first result in 15, we get that ∂A is the union of two closed sets in X, and is hence closed. □

Definition 17 (Subspaces)

Given $(X, \mathcal{T})$ a topological space with a non-empty subset A, the subspace topology on A is defined as

$$\mathcal{T}_A := \{A \cap U : U \in \mathcal{T}\}$$