18.085 Solutions Quiz 1 - Summer Session 2011

July 1, 2011

Problem 1. Consider the second order equation $-u''(x) = \delta(x - \frac{1}{2})$.
(a) Find the solution for the fixed-free homogeneous boundary conditions $u(0) = u'(1) = 0$.
(b) Find the solution for the boundary conditions $u(0) = 1, u'(1) = -2$.
(c) Write the difference equation with $h = \frac{1}{4}$ for the equation with the boundary conditions as in (a), and solve it. What do the fixed-free boundary conditions translate into? Does your solution vector $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ match the continuous solution from (a)?

(a) The solution to the differential equation has the form $u = -R(x - \frac{1}{2}) + Ax + B$. From the initial $u(0) = B = 0$ and $U'(0) = 1 + A = 0$. So $u = -R(x - \frac{1}{2}) + x$.

(b) For the boundary conditions $u(0) = 1$ and $u'(1) = -2$ the solution is $u = -R(x - \frac{1}{2}) - x + 1$.

(c) The difference equation is the following (for $h = \frac{1}{4}$):

$$\frac{1}{1/16}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{1/4}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and then $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/2 \end{bmatrix}$. This matches the solution from (a) since $u(1/4) = 0 + 1/4, u(1/2) = 0 + 1/2, u(3/4) = -1/4 + 3/4 = 1/2$.

Problem 2. Consider the following system of 3 springs and 2 masses:
(a) Write down the matrix A linking mass displacements to spring elongations. Find the stiffness matrix $K = A^T CA$.
(b) Assuming $c_1, c_2, c_3 > 0$, use two different tests to show that K is positive definite.
(c) Consider the following vector of external forces $f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Solve $Ku = f$. What is the elongation of spring c_1? How do you explain your answer physically?

(a) A is the matrix $\begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$. The stiffness matrix is $A^T CA = \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 + c_2 + c_3 \end{bmatrix}$.

(b) First test: check principal minors are positive: $c_1 > 0$ and $det(A^T CA) = c_1(c_1 + c_2 + c_3) - c_1^2 = c_1(c_2 + c_3) > 0$.
Second test: check pivots are positive: row reducing we obtain $\begin{bmatrix} c_1 & -c_1 \\ 0 & c_2 + c_3 \end{bmatrix}$.

(c) The equation is $\begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 + c_2 + c_3 \end{bmatrix} u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $u = \frac{1}{c_1(c_2+c_3)} \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} = \frac{1}{c_2+c_3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The elongation of the spring is $e_1 = u_2 - u_1 = 0$ so m_1 and m_2 move together.

Problem 3. Consider the spring-mass system in Problem 2, with $m_1 = m_2 = 1$ and $c_1 = c_2 = 2, c_3 = 1$, but now with masses moving in time rather than in equilibrium. We have seen that the displacements $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ are solutions to the second order vector differential equation $Mu'' + Ku = 0$.
(a) Find the eigenvalues λ_1, λ_2 and corresponding eigenvectors v_1, v_2 of the matrix $M^{-1}K$.
(b) Using (a), find the solution to the above equation $Mu'' + Ku = 0$ satisfying $u(0) = 0, u'(0) = v_1$. Then find the solution satisfying $u(0) = 0, u'(0) = v_2$.
(c) Each of the solutions in (b) describes a motion of masses in time. Explain what happens physically: Which mass is moving faster? Are the masses moving in the same or opposite directions? What are the oscillation frequencies?

(a) $A = M^{-1}K = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$, to find the eigenvalues: $\lambda_1 + \lambda_2 = trace(A) = 7$, $\lambda_1\lambda_7 = det(A) = 6$. Solving these equations $\lambda_1 = 1, \lambda_2 = 6$. The respective eigenvectors are $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(b) The solution satisfying $u(0) = 0, u'(0) = v_1$ is $u_1 = \sin(\sqrt{\lambda_1}t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \sin(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The solution satisfying $u(0) = 0, u'(0) = v_2$ is $u_2 = \sin(\sqrt{6}t) \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(c) For u_1, m_1 moves two times faster than m_2 and they move in the same direction (components of v_1 have same sign), the frequency of oscillation is $\sqrt{\lambda_1} = 1$. For u_2, m_1 moves two times slower than m_2, they move in opposite directions, the frequency of oscillation is $\sqrt{\lambda_2} = \sqrt{6}$.