18.085 HOMEWORK 1 SOLUTIONS - SUMMER SESSION 2011

1.1.5. We have $K_2^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and $\det K_5 = 6$ (we have seen, more generally, that $\det K_n = n+1$). Using Matlab, we also find

$$K_5^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

1.1.12. We perform row elimination on C_4 as follows:

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \overset{l_{21}=l_{41}=-\frac{1}{2}}{\longrightarrow} \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & \frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & -1 & 2 & -1 \\ 0 & -\frac{1}{2} & -1 & \frac{3}{2} \end{bmatrix} \overset{l_{32}=-\frac{2}{3}, l_{42}=-\frac{1}{3}}{\longrightarrow}$$
$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & \frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & \frac{4}{3} & -\frac{4}{3} \\ 0 & 0 & -\frac{4}{3} & \frac{4}{3} \end{bmatrix} \overset{l_{43}=-1}{\longrightarrow} \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & \frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & \frac{4}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last entry of U is 0 because C_4 is singular. The last column of U has new non-zeros because the top right -1 from C_4 propagates downwards under row operations.

1.1.26. We have

$$A = toeplitz(v) = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} & v_n \\ v_2 & v_1 & \ddots & & v_{n-1} \\ \vdots & \ddots & \ddots & & \vdots \\ v_{n-1} & & & \ddots & v_2 \\ v_n & v_{n-1} & \cdots & v_2 & v_1 \end{bmatrix}.$$

This matrix is circulant if and only if $A_{21} = A_{1n}, A_{31} = A_{2n}$,etc. i.e. if and only if $v_2 = v_n, v_3 = v_{n-1}$, etc.

1.2.7. By using Taylor expansion, we have

$$\begin{aligned}
u(x+2h) &= u(x) + 2hu'(x) + \frac{1}{2}4h^2u''(x) + \frac{1}{6}8h^3u'''(x) + \frac{1}{24}16h^4u''''(x) + \frac{1}{120}32h^5u'''''(x) + \ldots \\
u(x+h) &= u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u'''(x) + \frac{1}{24}h^4u''''(x) + \frac{1}{120}h^5u'''''(x) + \ldots \\
u(x-h) &= u(x) - hu'(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u'''(x) + \frac{1}{24}h^4u''''(x) - \frac{1}{120}h^5u'''''(x) + \ldots \\
u(x-2h) &= u(x) - 2hu'(x) + \frac{1}{2}4h^2u''(x) - \frac{1}{6}8h^3u'''(x) + \frac{1}{24}16h^4u''''(x) - \frac{1}{120}32h^5u'''''(x) + \ldots
\end{aligned}$$

A straightforward calculation implies

$$\frac{-u(x+2h) + 8u(x+h) - 8u(x-h) + u(x-2h)}{12h} = u'(x) - \frac{1}{30}h^4u'''''(x) + \ldots$$

Thus, $b = -1/30$. In particular, for $u = 1, u = x^2$ and $u = x^4$, we have

$$\frac{-u_2 + 8u_1 - 8u_{-1} + u_{-2}}{12h} = \frac{du}{dx}$$

without any error term (since in this case, 5th and higher order derivatives all vanish).

1.2.14.(a) Integrating $-u'' = 12x^2$ twice, we get $u = -x^4 + Cx + D$. Now the boundary conditions $u'(0) = u(1) = 0$ imply $C = 0, D = 1$. Hence $u = 1 - x^4$.

(b) For $n = 7$, the following Matlab code produces the discrete solution vector:

```
T = toeplitz([2 -1 zeros(1,5)]); T(1,1) = 1;
f = 12*((1:7)'*(1/8)).^2;
u = 1/64 *T\f
u =
0.9844
0.9814
0.9668
0.9258
0.8379
0.6768
0.4102
```

At the center point $x = \frac{1}{2}$, we have $u = 1 - (\frac{1}{2})^4 = 0.9375$ while the discrete solution has $u_4 = 0.9258$, which is off by 0.0127. One can do the same procedure for $n = 3$,$n = 15$, and find the corresponding errors to be 0.0469 and 0.0029 respectively. The errors seem proportional to h^2.

1.2.18. The general solution to $\frac{d^2u}{dx^2} = x$ is $u = \frac{1}{6}x^3 + Bx + C$. The boundary conditions $u(0) = u(1) = 0$ imply $C = 0, B = -\frac{1}{6}$, i.e. $u(x) = \frac{x^3-x}{6}$.

Hence $u(0.2) = -0.032, u(0.4) = -0.056, u(0.6) = -0.064, u(0.8) = -0.048$. The finite difference equation is:

$$-\frac{1}{h^2}K_4 \begin{bmatrix} u_1 \\ \vdots \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.2 \\ \vdots \\ 0.8 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} u_1 \\ \vdots \\ u_4 \end{bmatrix} = \frac{-1}{125}K_4^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{-1}{125} \cdot \frac{1}{5} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -.032 \\ -.056 \\ -.064 \\ -.048 \end{bmatrix},$$

which perfectly matches the values of the actual solution.

1.3.8. We find $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. We have

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \overset{l_{21} = \frac{1}{2}}{\longrightarrow} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 1 & 2 \end{bmatrix} \overset{l_{32} = \frac{2}{3}}{\longrightarrow} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

hence $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$.

1.3.10. Subtracting row 1 from all the other rows, we obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus there is only one (non-zero) pivot, namely 1. The determinant is 0. Trying `eig(ones(4))` in Matlab reveals the eigenvalues 0,0,0,4. Since it is a rank1 matrix, it can be factored as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$