18.085 HOMEWORK 2 SOLUTIONS - SUMMER SESSION 2011

1.4.6. Integrating $-u'' = \delta(x-a)$ from 0 to 1, we obtain $u'(0) - u'(1) = 1$. Thus, the boundary condition $u'(0) = u'(1)$ can never be met.

1.4.7. Integrating $-u'' = \delta(x-\frac{1}{3}) - \delta(x-\frac{2}{3})$ twice, we obtain $u(x) = R(x-\frac{2}{3}) - R(x-\frac{1}{3}) + Cx + D$. Now $u'(0) = 0$ and $u'(1) = 0$ imply $C = 0$. Since D can be anything, we have infinitely many solutions: $u(x) = R(x-\frac{2}{3}) - R(x-\frac{1}{3}) + D$.

1.4.8. From 1.4.7, we know u has the form $u = R(x-\frac{2}{3}) - R(x-\frac{1}{3}) + Cx + D$. The periodic boundary conditions $u(0) = u(1), u'(0) = u'(1)$ imply $D = -\frac{1}{3} + C + D$, i.e. $C = -\frac{1}{3}$. A particular solution can be obtained by setting $D = 0$: $u_{part} = R(x-\frac{2}{3}) - R(x-\frac{1}{3}) - \frac{1}{3}x$. u_{null} is the general solution to $-u'' = 0$ with periodic boundary conditions, i.e. $u_{null} = D$ (constant).

1.4.12. The complete solution to $u'''' = \delta(x)$ is $u(x) = C(x) + a + bx + cx^2 + dx^3$ ($u_{null} = a + bx + cx^2 + dx^3$ is the general solution to $u'''' = 0$). We have

$$u(x) = \begin{cases} a + bx + cx^2 + dx^3 & x \leq 0 \\ a + bx + cx^2 + dx^3 + \frac{1}{6}x^3 & x \geq 0 \end{cases} \Longrightarrow \begin{array}{c} u(1) = a + b + c + d + \frac{1}{6} \\ u(-1) = a - b + c - d \\ u''(1) = 2c + (6d+1) \\ u''(-1) = 2c - 6d \end{array}$$

Now the boundary conditions $u(1) = u(-1) = u''(1) = u''(-1) = 0$ imply (after some computations) $a = \frac{1}{6}, b = 0, c = -\frac{1}{4}, d = -\frac{1}{12}$.

1.5.2. We have $2\sin\pi h - \sin 2\pi h = 2\sin\pi h - 2\sin\pi h\cos\pi h = 2\sin\pi h(1-\cos\pi h)$. Thus, $\lambda = 2 - 2\cos\pi h$.

1.5.12. We have $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, S = \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow$ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

1.5.23. The eigenvalues of $A = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$ are $\lambda_1 = 4, \lambda_2 = 1$. Solving $(A - 4I)x = 0$ we find an eigenvector corresponding to λ_1: $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Similarly, for λ_2 we get $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The general solution to $\frac{du}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} u$ is

$u = c_1e^{\lambda_1 t}x_1 + c_2e^{\lambda_2 t}x_2 = \begin{bmatrix} c_1e^{4t} + c_2e^t \\ -c_2e^t \end{bmatrix}$. The initial condition $u(0) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ would imply $c_2 = 2, c_1 = 3$ (for our choice of x_1, x_2).

1.5.25. The system can be written as

$$\frac{d}{dt}\begin{bmatrix} r \\ w \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} r \\ w \end{bmatrix}.$$

A is the coefficient matrix $A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$. Its eigenvalues λ_1, λ_2 satisfy $\lambda_1 + \lambda_2 = trA = 7, \lambda_1\lambda_2 = \det A = 10$, hence $\lambda_1 = 2, \lambda_2 = 5$. We find corresponding eigenvectors $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Hence the solution has the form $\begin{bmatrix} r \\ w \end{bmatrix} = c_1e^{\lambda_1 t}x_1 + c_2e^{\lambda_2 t}x_2 = \begin{bmatrix} c_1e^{2t} + 2c_2e^{5t} \\ 2c_1e^{2t} + c_2e^{5t} \end{bmatrix}$. The initial condition

$\begin{bmatrix} r(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \end{bmatrix}$ implies $c_1 = c_2 = 10$, hence $\begin{bmatrix} r(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} 10e^{2t} + 20e^{5t} \\ 20e^{2t} + 10e^{5t} \end{bmatrix}$. When t is large, the e^{5t} term dominates e^{2t}, hence $\frac{r(t)}{w(t)} \approx \frac{20e^{5t}}{10e^{5t}} = \frac{2}{1}$.

1.6.9. We perform row elimination on the given symmetric matrices:

$$\begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \xrightarrow{l_{21} = b} \begin{bmatrix} 1 & b \\ 0 & 9 - b^2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \xrightarrow{l_{21} = 2} \begin{bmatrix} 2 & 4 \\ 0 & c - 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c - 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

The matrices are positive definite if and only if their pivots are all positive, i.e. if $-3 < b < 3$ and $c > 8$.

1.6.13. The entries of S can be found as the coefficients of the polynomial on the right hand side: $s_{ii} = coeff(x_i^2), s_{ij} = \frac{1}{2}coeff(x_i x_j)$. Hence $S = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$. All rows are multiples of each other, hence there is only one (non-zero) pivot in the top left corner of S: 4. In particular, S has rank 1, hence S has only one non-zero eigenvalue which can be found from the trace of S: $\lambda_1 = 24$. The determinant is 0. The rank of S can also be read from the right hand side as the number of squared terms (one).