18.085 HOMEWORK 3 SOLUTIONS - SUMMER SESSION 2011

1.6.28. Solving $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} x_i = \lambda_i x_i$, we find the eigenvectors of the KKT matrix. The following eigenvectors have unit length: $x_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, x_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, x_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Hence, one expects

$$\begin{bmatrix} w_1 & w_2 & u \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix} = 1\left(\frac{w_1+w_2}{\sqrt{2}}\right)^2 + 2\left(\frac{w_1-w_2-u}{\sqrt{3}}\right)^2 - 1\left(\frac{w_1-w_2+2u}{\sqrt{6}}\right)^2.$$

This can identity can be checked directly.

1.7.18. We have $AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, whose eigenvalues satisfy $\lambda_1 + \lambda_2 = 4, \lambda_1\lambda_2 = 3$. Hence they are $\lambda_1 = 3, \lambda_2 = 1$. The singular values of A are the square roots of these: $\sigma_1 = \sqrt{3}, \sigma_2 = 1$.

Solving $AA^T x_i = \lambda_i x_i$, we find the following unit eigenvectors of AA^T: $x_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which gives $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$. We know that λ_1, λ_2 are the non-zero eigenvalues of $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Hence the third eigenvalue $\lambda_3 = 0$. One finds the corresponding unit eigenvectors of A^TA: $v_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, which give $V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

We compute

$$\begin{aligned} U\Sigma V^T &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A. \end{aligned}$$

1.7.20. $Av_1 = u_1, ..., Av_n = u_n$ imply $AV = U$, where $V = \begin{bmatrix} v_1 & ... & v_n \end{bmatrix}, U = \begin{bmatrix} u_1 & ... & u_n \end{bmatrix}$. These U, V are orthogonal matrices, since $u_1, ..., u_n$ and $v_1, ..., v_n$ are orthonormal bases of $\mathbb{R}^n$. Hence, multiplying both sides of $AV = U$ by $V^{-1} = V^T$ on the right, we obtain $A = UV^T = UI_nV^T$. This is the singular decomposition of A, and the singular values are all 1.

2.1.5. For the fixed-fixed case with 3 masses and 4 springs, the difference matrix is $A = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{bmatrix}$ and the stiffness matrix is $K = A^T CA = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix} = c \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. The force vector coming from gravity is $f = \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix}$. Hence the mass displacements are

$$u = K^{-1} f = \frac{mg}{4c} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{mg}{2c} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}.$$

The reaction forces in springs are given by $w = CAu = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \frac{mg}{2} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3mg}{2} \\ \frac{mg}{2} \\ -\frac{mg}{2} \\ \frac{-3mg}{2} \end{bmatrix}$.
The reaction force on top of spring 1 is $w_i = \frac{3mg}{2}$ and on the bottom of spring 4, $w_4 = -\frac{3mg}{2}$. $|w_1|+|w_2|$ balances the total force from gravity $3mg$.

2.1.6. For the fixed-free case, we have

$$K = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} = \begin{bmatrix} c_2+1 & -c_2 & 0 \\ -c_2 & c_2+1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

For $c_2 = 10$, we find $u = K^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.2 \\ 4.2 \end{bmatrix}$. For $c_2 = 100$, $u = K^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.02 \\ 4.02 \end{bmatrix}$. If we let $c_2 \to \infty$, we see that the displacements tend to $\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$, which makes sense, because masses 1 and 2 would essentially be glued together.

2.1.7. For the fixed-fixed case, we have

$$K = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix}$$

With $c_1 = c_3 = c_4 = 1, c_2 = 0$ this becomes $K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, which is invertible and $u = K^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. Physically, weakening spring 2 breaks the problem into two subproblems: a fixed-free with mass 1 hanging on spring 1, and a free-fixed with masses 2,3 and springs 3,4.

2.2.5(a) The derivative of $\|u(t)\|^2$ is $2u_1u_1' + 2u_2u_2' + 2u_3u_3' = 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) = 0$. Hence $\|u(t)\|^2$ is constant, and in particular, $\|u(t)\|^2 = \|u(0)\|^2$.

2.2.6. Equation (24) reads

$$U_{n+1} - \frac{\Delta t}{2} AU_{n+1} = U_n + \frac{\Delta t}{2} AU_n.$$

Taking dot products of both sides with $U_{n+1} + U_n$ we obtain

$$\begin{aligned}(U_{n+1} - \frac{\Delta t}{2} AU_{n+1}) \cdot (U_{n+1} + U_n) &= (U_n + \frac{\Delta t}{2} AU_n) \cdot (U_{n+1} + U_n) \Rightarrow \\ U_{n+1} \cdot U_{n+1} - \frac{\Delta t}{2} AU_{n+1} \cdot U_{n+1} + U_{n+1} \cdot U_n - \frac{\Delta t}{2} AU_{n+1} \cdot U_n &= \\ U_n \cdot U_{n+1} + \frac{\Delta t}{2} AU_n \cdot U_{n+1} + U_n \cdot U_n + \frac{\Delta t}{2} AU_n \cdot U_n\end{aligned}$$

But $A^T = -A$ implies that for any vectors u, v, we have $Au \cdot v = u \cdot A^T v = -u \cdot Av = -Av \cdot u$. In particular, $Au \cdot u = -u \cdot Au$, which implies $Au \cdot u = 0$.

Hence the above equation simplifies to

$$U_{n+1} \cdot U_{n+1} = U_n \cdot U_n \Rightarrow \|U_{n+1}\|^2 = \|U_n\|^2 .$$

2.2.8. Using Google's calculator, one finds $(1 + h^2)^{32} \approx 3.355205$. Using $\lim_{x \to 0} (1 + x)^{1/x} = e$, we find

$$\lim_{h \to 0} (1 + h^2)^{2\pi/h} = \lim_{h \to 0} \left((1 + h^2)^{1/h^2} \right)^{2\pi h} = \lim_{h \to 0} e^{2\pi h} = 1.$$

A backward Euler step is given by

$$\begin{aligned}U_{n+1} &= U_n + hV_{n+1} \\ V_{n+1} &= V_n - hU_{n+1}\end{aligned}$$

In other words, $U_n = U_{n+1} - hV_{n+1}, V_n = V_{n+1} + hU_{n+1}$, i.e. (U_n, V_n) is related to (U_{n+1}, V_{n+1}) by a Forward Euler step with h replaced by $-h$. Hence

$$U_n^2 + V_n^2 = (1 + (-h)^2)(U_{n+1}^2 + V_{n+1}^2) \Rightarrow U_{n+1}^2 + V_{n+1}^2 = \frac{1}{1 + h^2}(U_n^2 + V_n^2).$$