18.085 HOMEWORK 4 SOLUTIONS - SUMMER SESSION 2011

2.3.7. The least squares problem for fitting a line $y = C+Dx$ through the points (0,4),(1,1),(2,0),(3,1) is $A^TA\begin{bmatrix} C \\ D \end{bmatrix} = A^Tb$, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. The equation becomes $\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$. Solving, we obtain $(C, D) = (3, -1)$.

2.3.10. We are trying to solve $C = 4, C = 1, C = 0, C = 1$, i.e. $A[C] = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Hence $A^TA[C] = A^T\begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow 4C = 6 \Rightarrow C = \frac{3}{2}$. Thus the best horizontal line is $y = \frac{3}{2}$.

2.3.12. For the closest parabola $y = C + Dx + Ex^2$ to b at the points $x = (0, 1, 2, 3)$, the unsolvable equations are $A\begin{bmatrix} C \\ D \\ E \end{bmatrix} = b$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Using Matlab, we find the least squares solution $\begin{bmatrix} C \\ D \\ E \end{bmatrix} = A^TA\backslash A^Tb = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$. Fitting instead to a cubic $y = C + Dx + Ex^2 + Fx^3$ amounts to solving $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. This can be solved exactly, since the coefficient matrix is invertible (as can be checked with Matlab, for example). The error vector e should be 0 in this case, since the problem can be solved exactly.

2.4.5. If there is an edge $i \to j$ in the graph, then some row of A has the form $[0...0\ -1...0\ 1...0]$ with -1 and 1 in positions i and j respectively. Then the corresponding equation of the system $Au = 0$ reads $-u_i + u_j = 0 \Rightarrow u_i = u$. For any $i \neq j$, since the graph is connected, there exists a path $i \to i_1 \to ... \to i_l \to j$. By the earlier argument, $u_i = u_{i_1}, u_{i_1} = u_{i_2}, ..., u_{i_l} = u_j$, hence $u_i = u_j$. Therefore, the solutions to $Au = 0$ are $\begin{bmatrix} C \\ \vdots \\ C \end{bmatrix}$.

2.4.6. The ii entry of A^TA is $\sum_{k=1}^n a_{ki}a_{ki} = \sum_{k-\text{adjacent to } i}(\pm 1)^2 = \deg i$. Hence $tr(A^TA) = \sum_{i=1}^n \deg i = 2m$, because each edge is counted twice.

2.4.7. We have $A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, hence

$$K = A^T \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix} A = \begin{bmatrix} c_1 & 0 & 0 & -c_1 \\ 0 & c_2 & 0 & -c_2 \\ 0 & 0 & c_3 & -c_3 \\ -c_1 & -c_2 & -c_3 & c_1+c_2+c_3 \end{bmatrix}$$

Grounding node 4 means deleting the 4th column of A, hence deleting the last row and column of K. Thus, $K_{red} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$, $\det K = c_1c_2c_3(c_1+c_2+c_3)$.

2.4.9. We have $A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$, $A^TA = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$. Removing row 5 and column 5 of A^TA gives $(A^TA)_{red} = K = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = T_4$, hence $K^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Using Matlab, one finds the eigenvalues of K to be $1, 0.1206, 2.3473, 3.5321$, and $\det K = 1$.

2.4.13. It is straight-forward to list all the trees. There are 3 for the triangle and 8 for the square graph.

2.4.17. **(a)** The number of zeros among the entries of the Laplacian A^TA equals the number of (ordered) pairs of nodes not connected by an edge. There are $m = 12$ edges in the 3 by 3 grid, so the number of zeros is $81 - n - 2m = 48$.

(b) The diagonal contains the node degrees: $D = diag(2,3,2,3,4,3,2,3,2)$.

(c) The middle row corresponds to the center node, which is connected to 4 other nodes. Hence, $d_{55} = 4$, and there are four -1's in W, one for each edge coming out of the center node.

2.4.18. The K matrix for the 3 by 3 grid is

$$K = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Grounding node (3,3) has the effect of erasing the last row and columns of K. The voltages u satisfy $K_{red}u = f_{red}$, where $f_{red} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Solving in Matlab we find $u = K_{red}^{-1} f_{red} = \begin{bmatrix} 1.5 \\ 1 \\ .75 \\ 1 \\ .75 \\ .5 \\ .75 \\ .5 \end{bmatrix}$.
(u is just the first column of K_{red}^{-1}). The grid resistance between the far corners is thus $u(1,1) = 1.5$.