Calculus

Harvard-MIT Math Tournament
February 27, 1999

1. Find all twice differentiable functions $f(x)$ such that $f''(x) = 0$, $f(0) = 19$, and $f(1) = 99$.

2. A rectangle has sides of length $\sin x$ and $\cos x$ for some x. What is the largest possible area of such a rectangle?

3. Find

$$\int_{-4\pi\sqrt{2}}^{4\pi\sqrt{2}} \left(\frac{\sin x}{1+x^4} + 1\right) dx.$$

4. f is a continuous real-valued function such that $f(x+y) = f(x)f(y)$ for all real x, y. If $f(2) = 5$, find $f(5)$.

5. Let $f(x) = x + \frac{1}{2x+\frac{1}{2x+\frac{1}{2x+\ddots}}}$ for $x > 0$. Find $f(99)f'(99)$.

6. Evaluate $\frac{d}{dx}(\sin x - \frac{4}{3}\sin^3 x)$ when $x = 15$.

7. If a right triangle is drawn in a semicircle of radius 1/2 with one leg (not the hypotenuse) along the diameter, what is the triangle's maximum possible area?

8. A circle is randomly chosen in a circle of radius 1 in the sense that a point is randomly chosen for its center, then a radius is chosen at random so that the new circle is contained in the original circle. What is the probability that the new circle contains the center of the original circle?

9. What fraction of the Earth's volume lies above the 45 degrees north parallel? You may assume the Earth is a perfect sphere. The volume in question is the smaller piece that we would get if the sphere were sliced into two pieces by a plane.

10. Let A_n be the area outside a regular n-gon of side length 1 but inside its circumscribed circle, let B_n be the area inside the n-gon but outside its inscribed circle. Find the limit as n tends to infinity of $\frac{A_n}{B_n}$.