Oral Event

Harvard-MIT Math Tournament
February 27, 1999

1. [25] Start with an angle of $60°$ and bisect it, then bisect the lower $30°$ angle, then the upper $15°$ angle, and so on, always alternating between the upper and lower of the previous two angles constructed. This process approaches a limiting line that divides the original $60°$ angle into two angles. Find the measure (degrees) of the smaller angle.

2. [25] Alex, Pei-Hsin, and Edward got together before the contest to send a mailing to all the invited schools. Pei-Hsin usually just stuffs the envelopes, but if Alex leaves the room she has to lick them as well and has a 25% chance of dying from an allergic reaction before he gets back. Licking the glue makes Edward a bit psychotic, so if Alex leaves the room there is a 20% chance that Edward will kill Pei-Hsin before she can start licking envelopes. Alex leaves the room and comes back to find Pei-Hsin dead. What is the probability that Edward was responsible?

3. [30] If x, y, and z are distinct positive integers such that $x^2 + y^2 = z^3$, what is the smallest possible value of $x + y + z$.

4. [35] Evaluate $\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$, where $\cos\theta = \frac{1}{5}$.

5. [45] Let, r be the inradius of triangle ABC. Take a point D on side BC, and let r_1 and r_2 be the inradii of triangles ABD and ACD. Prove that r, r_1, and r_2 can always be the side lengths of a triangle.

6. [45] You want to sort the numbers 5 4 3 2 1 using block moves. In other words, you can take any set of numbers that appear consecutively and put them back in at any spot as a block. For example, *6 5 3* 4 2 1 $\to$ 4 2 *6 5 3* 1 is a valid block move for 6 numbers. What is the minimum number of block moves necessary to get 1 2 3 4 5?

7. [55] Evaluate $\sum_{n=1}^{\infty} \frac{n^5}{n!}$.

8. [55] What is the smallest square-free composite number that can divide a number of the form $4242\ldots42 \pm 1$?

9. [60] You are somewhere on a ladder with 5 rungs. You have a fair coin and an envelope that contains either a double-headed coin or a double-tailed coin, each with probability 1/2. Every minute you flip a coin. If it lands heads you go up a rung, if it lands tails you go down a rung. If you move up from the top rung you win, if you move down from the bottom rung you lose. You can open the envelope at any time, but if you do then you must immediately flip that coin once, after which you can use it or the fair coin whenever you want. What is the best strategy (i.e. on what rung(s) should you open the envelope)?

10. [75] A, B, C, D, and E are relatively prime integers (i.e., have no single common factor) such that the polynomials $5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E$ and $10Ax^3 + 6Bx^2 + 3Cx + D$ together have 7 distinct integer roots. What are all possible values of A? Your team has been given a sealed envelope that contains a hint for this problem. If you open the envelope, the value of this problem decreases by 20 points. To get full credit, give the sealed envelope to the judge before presenting your solution.