Advanced Topics Test

Harvard-MIT Math Tournament

March 3, 2001

1. Find $x - y$, given that $x^4 = y^4 + 24$, $x^2 + y^2 = 6$, and $x + y = 3$.

2. Find $\log_n\left(\frac{1}{2}\right)\log_{n-1}\left(\frac{1}{3}\right)\cdots\log_2\left(\frac{1}{n}\right)$ in terms of n.

3. Calculate the sum of the coefficients of $P(x)$ if $(20x^{27} + 2x^2 + 1)P(x) = 2001x^{2001}$.

4. Boris was given a Connect Four game set for his birthday, but his color-blindness makes it hard to play the game. Still, he enjoys the shapes he can make by dropping checkers into the set. If the number of shapes possible modulo (horizontal) flips about the vertical axis of symmetry is expressed as $9(1 + 2 + \cdots + n)$, find n. (Note: the board is a vertical grid with seven columns and eight rows. A checker is placed into the grid by dropping it from the top of a column, and it falls until it hits either the bottom of the grid or another checker already in that column. Also, $9(1 + 2 + \cdots + n)$ is the number of shapes possible, with two shapes that are horizontal flips of each other counted as one. In other words, the shape that consists solely of 3 checkers in the rightmost row and the shape that consists solely of 3 checkers in the leftmost row are to be considered the same shape.)

5. Find the 6-digit number beginning and ending in the digit 2 that is the product of three consecutive even integers.

6. There are two red, two black, two white, and a positive but unknown number of blue socks in a drawer. It is empirically determined that if two socks are taken from the drawer without replacement, the probability they are of the same color is $\frac{1}{5}$. How many blue socks are there in the drawer?

7. Order these four numbers from least to greatest: $5^{56}, 10^{51}, 17^{35}, 31^{28}$.

8. Find the number of positive integer solutions to $n^x + n^y = n^z$ with $n^z < 2001$.

9. Find the real solutions of $(2x + 1)(3x + 1)(5x + 1)(30x + 1) = 10$.

10. Alex picks his favorite point (x, y) in the first quadrant on the unit circle $x^2 + y^2 = 1$, such that a ray from the origin through (x, y) is θ radians counterclockwise from the positive x-axis. He then computes $\cos^{-1}\left(\frac{4x+3y}{5}\right)$ and is surprised to get θ. What is $\tan(\theta)$?