General Test, First Half

Harvard-MIT Math Tournament

March 3, 2001

1. What is the last digit of $17^{103} + 5$?

2. Find $x + y$, given that $x^2 - y^2 = 10$ and $x - y = 2$.

3. There are some red and blue marbles in a box. We are told that there are twelve more red marbles than blue marbles, and we experimentally determine that when we pick a marble randomly we get a blue marble one quarter of the time. How many marbles are there in the box?

4. Find $a + b + c + d + e$ if

$$\begin{gathered} 3a + 2b + 4d = 10, \\ 6a + 5b + 4c + 3d + 2e = 8, \\ a + b + 2c + 5e = 3, \\ 2c + 3d + 3e = 4, \text{ and} \\ a + 2b + 3c + d = 7. \end{gathered}$$

5. What is the sum of the coefficients of the expansion $(x + 2y - 1)^6$?

6. A right triangle has a hypotenuse of length 2, and one of its legs has length 1. The altitude to its hypotenuse is drawn. What is the area of the rectangle whose diagonal is this altitude?

7. Find $(x + 1)(x^2 + 1)(x^4 + 1)(x^8 + 1) \cdots$, where $|x| < 1$.

8. How many times does 24 divide into 100!?

9. Boris was given a Connect Four game set for his birthday, but his color-blindness makes it hard to play the game. Still, he enjoys the shapes he can make by dropping checkers into the set. If the number of shapes possible modulo (horizontal) flips about the vertical axis of symmetry is expressed as $9(1 + 2 + \cdots + n)$, find n. (Note: the board is a vertical grid with seven columns and eight rows. A checker is placed into the grid by dropping it from the top

of a column, and it falls until it hits either the bottom of the grid or another checker already in that column. Also, $9(1 + 2 + \cdots + n)$ is the number of shapes possible, with two shapes that are horizontal flips of each other counted as one. In other words, the shape that consists solely of 3 checkers in the rightmost row and the shape that consists solely of 3 checkers in the leftmost row are to be considered the same shape.)

10. Find the 6-digit number beginning and ending in the digit 2 that is the product of three consecutive even integers.