Harvard-MIT Math Tournament
March 17, 2002
Individual Subject Test: **Advanced Topics**

1. Eight knights are randomly placed on a chessboard (not necessarily on distinct squares). A knight on a given square attacks all the squares that can be reached by moving either (1) two squares up or down followed by one squares left or right, or (2) two squares left or right followed by one square up or down. Find the probability that every square, occupied or not, is attacked by some knight.

2. A certain cafeteria serves ham and cheese sandwiches, ham and tomato sandwiches, and tomato and cheese sandwiches. It is common for one meal to include multiple types of sandwiches. On a certain day, it was found that 80 customers had meals which contained both ham and cheese; 90 had meals containing both ham and tomatoes; 100 had meals containing both tomatoes and cheese. 20 customers' meals included all three ingredients. How many customers were there?

3. How many four-digit numbers are there in which at least one digit occurs more than once?

4. Two fair coins are simultaneously flipped. This is done repeatedly until at least one of the coins comes up heads, at which point the process stops. What is the probability that the other coin also came up heads on this last flip?

5. Determine the number of subsets S of $\{1, 2, 3, \ldots, 10\}$ with the following property: there exist integers $a < b < c$ with $a \in S, b \notin S, c \in S$.

6. In how many ways can the numbers $1, 2, \ldots, 2002$ be placed at the vertices of a regular 2002-gon so that no two adjacent numbers differ by more than 2? (Rotations and reflections are considered distinct.)

7. A manufacturer of airplane parts makes a certain engine that has a probability p of failing on any given flight. Their are two planes that can be made with this sort of engine, one that has 3 engines and one that has 5. A plane crashes if more than half its engines fail. For what values of p do the two plane models have the same probability of crashing?

8. Given a 9×9 chess board, we consider all the rectangles whose edges lie along grid lines (the board consists of 81 unit squares, and the grid lines lie on the borders of the unit squares). For each such rectangle, we put a mark in every one of the unit squares inside it. When this process is completed, how many unit squares will contain an even number of marks?

9. Given that a, b, c are positive real numbers and $\log_a b + \log_b c + \log_c a = 0$, find the value of $(\log_a b)^3 + (\log_b c)^3 + (\log_c a)^3$.

10. One fair die is rolled; let a denote the number that comes up. We then roll a dice; let the sum of the resulting a numbers be b. Finally, we roll b dice, and let c be the sum of the resulting b numbers. Find the expected (average) value of c.