Harvard-MIT Math Tournament
March 17, 2002
Individual Subject Test: **Algebra**

1. Nine nonnegative numbers have average 10. What is the greatest possible value for their median?

2. p and q are primes such that the numbers $p + q$ and $p + 7q$ are both squares. Find the value of p.

3. Real numbers a, b, c satisfy the equations $a + b + c = 26, 1/a + 1/b + 1/c = 28$. Find the value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{a}{c} + \frac{c}{b} + \frac{b}{a}.$$

4. If a positive integer multiple of 864 is picked randomly, with each multiple having the same probability of being picked, what is the probability that it is divisible by 1944?

5. Find the greatest common divisor of the numbers $2002 + 2, 2002^2 + 2, 2002^3 + 2, \ldots$.

6. Find the sum of the even positive divisors of 1000.

7. The real numbers x, y, z, w satisfy

$$
\begin{aligned}
2x + y + z + w &= 1 \\
x + 3y + z + w &= 2 \\
x + y + 4z + w &= 3 \\
x + y + z + 5w &= 25.
\end{aligned}
$$

Find the value of w.

8. Determine the value of the sum

$$\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots + \frac{29}{14^2 \cdot 15^2}.$$

9. For any positive integer n, let $f(n)$ denote the number of 1's in the base-2 representation of n. For how many values of n with $1 \leq n \leq 2002$ do we have $f(n) = f(n + 1)$?

10. Determine the value of

$$2002 + \frac{1}{2}(2001 + \frac{1}{2}(2000 + \cdots + \frac{1}{2}(3 + \frac{1}{2} \cdot 2)) \cdots).$$