Harvard-MIT Math Tournament
March 17, 2002
Individual Subject Test: **Calculus**

1. Two circles have centers that are d units apart, and each has diameter $\sqrt{d}$. For any d, let $A(d)$ be the area of the smallest circle that contains both of these circles. Find $\lim_{d\to\infty} \frac{A(d)}{d^2}$.

2. Find $\lim_{h\to 0} \frac{x^2-(x+h)^2}{h}$.

3. We are given the values of the differentiable real functions f, g, h, as well as the derivatives of their pairwise products, at $x = 0$:

$$f(0) = 1;\ g(0) = 2;\ h(0) = 3;\ (gh)'(0) = 4;\ (hf)'(0) = 5;\ (fg)'(0) = 6.$$

Find the value of $(fgh)'(0)$.

4. Find the area of the region in the first quadrant $x > 0, y > 0$ bounded above the graph of $y = \arcsin(x)$ and below the graph of the $y = \arccos(x)$.

5. What is the minimum vertical distance between the graphs of $2 + \sin(x)$ and $\cos(x)$?

6. Determine the positive value of a such that the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices $(0, 0), (a, 0), (0, a^2 + 1)$, and $(a, a^2 + 1)$.

7. Denote by $\langle x\rangle$ the fractional part of the real number x (for instance, $\langle 3.2\rangle = 0.2$). A positive integer N is selected randomly from the set $\{1, 2, 3, \ldots, M\}$, with each integer having the same probability of being picked, and $\left\langle \frac{87}{303}N\right\rangle$ is calculated. This procedure is repeated M times and the average value $A(M)$ is obtained. What is $\lim_{M\to\infty} A(M)$?

8. Evaluate $\int_0^{(\sqrt{2}-1)/2} \frac{dx}{(2x+1)\sqrt{x^2+x}}$.

9. Suppose f is a differentiable real function such that $f(x) + f'(x) \leq 1$ for all x, and $f(0) = 0$. What is the largest possible value of $f(1)$? (Hint: consider the function $e^x f(x)$.)

10. A continuous real function f satisfies the identity $f(2x) = 3f(x)$ for all x. If $\int_0^1 f(x)\,dx = 1$, what is $\int_1^2 f(x)\,dx$?