1. [4] An (l, a)-*design* of a set is a collection of subsets of that set such that each subset contains exactly l elements and that no two of the subsets share more than a elements. How many (2,1)-designs are there of a set containing 8 elements?

2. [5] A *lattice point* in the plane is a point of the form (n, m), where n and m are integers. Consider a set S of lattice points. We construct the *transform* of S, denoted by S', by the following rule: the pair (n, m) is in S' if and only if any of $(n, m-1)$, $(n, m+1)$, $(n-1, m)$, $(n+1, m)$, and (n, m) is in S. How many elements are in the set obtained by successively transforming $\{(0, 0)\}$ 14 times?

3. [5] How many elements are in the set obtained by transforming $\{(0, 0), (2, 0)\}$ 14 times?

4. [4] How many ways are there of using diagonals to divide a regular 6-sided polygon into triangles such that at least one side of each triangle is a side of the original polygon and that each vertex of each triangle is a vertex of the original polygon?

5. [6] Two 4×4 squares are randomly placed on an 8×8 chessboard so that their sides lie along the grid lines of the board. What is the probability that the two squares overlap?

6. [±6] Find all values of x that satisfy $x = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ (be careful; this is tricky).

7. [5] A rubber band is 4 inches long. An ant begins at the left end. Every minute, the ant walks one inch along rightwards along the rubber band, but then the band is stretched (uniformly) by one inch. For what value of n will the ant reach the right end during the nth minute?

8. [7] Draw a square of side length 1. Connect its sides' midpoints to form a second square. Connect the midpoints of the sides of the second square to form a third square. Connect the midpoints of the sides of the third square to form a fourth square. And so forth. What is the sum of the areas of all the squares in this infinite series?

9. [4] Find all values of x with $0 \leq x < 2\pi$ that satisfy $\sin x + \cos x = \sqrt{2}$.

10. **[6]** The mathematician John is having trouble remembering his girlfriend Alicia's 7-digit phone number. He remembers that the first four digits consist of one 1, one 2, and two 3s. He also remembers that the fifth digit is either a 4 or 5. While he has no memory of the sixth digit, he remembers that the seventh digit is 9 minus the sixth digit. If this is all the information he has, how many phone numbers does he have to try if he is to make sure he dials the correct number?

11. **[7]** How many real solutions are there to the equation

$$||||x|-2|-2|-2| = ||||x|-3|-3|-3| \ ?$$

12. [±7] This question forms a three question multiple choice test. After each question, there are 4 choices, each preceded by a letter. Please write down your answer as the ordered triple (letter of the answer of Question #1, letter of the answer of Question #2, letter of the answer of Question #3). If you find that all such ordered triples are logically impossible, then write "no answer" as your answer. If you find more than one possible sets of answers, then provide all ordered triples as your answer.

When we refer to "the correct answer to Question X" it is the actual answer, not the letter, to which we refer. When we refer to "the letter of the correct answer to question X" it is the letter contained in parentheses that precedes the answer to which we refer.

You are given the following condition: No two correct answers to questions on the test may have the same letter.

Question 1. If a fourth question were added to this test, and if the letter of its correct answer were (C), then:

(A) This test would have no logically possible set of answers.
(B) This test would have one logically possible set of answers.
(C) This test would have more than one logically possible set of answers.
(D) This test would have more than one logically possible set of answers.

Question 2. If the answer to Question 2 were "Letter (D)" and if Question 1 were not on this multiple-choice test (still keeping Questions 2 and 3 on the test), then the letter of the answer to Question 3 would be:

(A) Letter (B)
(B) Letter (C)
(C) Letter (D)
(D) Letter (A)

Question 3. Let $P_1 = 1$. Let $P_2 = 3$. For all $i > 2$, define $P_i = P_{i-1}P_{i-2} - P_{i-2}$. Which is a factor of P_{2002}?

(A) 3
(B) 4
(C) 7
(D) 9

13. [7] A *domino* is a 1-by-2 or 2-by-1 rectangle. A *domino tiling* of a region of the plane is a way of covering it (and only it) completely by nonoverlapping dominoes. For instance, there is one domino tiling of a 2-by-1 rectangle and there are 2 tilings of a 2-by-2 rectangle (one consisting of two horizontal dominoes and one consisting of two vertical dominoes). How many domino tilings are there of a 2-by-10 rectangle?

14. [7] An *omino* is a 1-by-1 square or a 1-by-2 horizontal rectangle. An *omino tiling* of a region of the plane is a way of covering it (and only it) by ominoes. How many omino tilings are there of a 2-by-10 horizontal rectangle?

15. [7] How many sequences of 0s and 1s are there of length 10 such that there are no three 0s or 1s consecutively anywhere in the sequence?

16. [5] Divide an m-by-n rectangle into mn nonoverlapping 1-by-1 squares. A *polyomino* of this rectangle is a subset of these unit squares such that for any two unit squares S, T in the polyomino, either

(1) S and T share an edge or

(2) there exists a positive integer n such that the polyomino contains unit squares $S_1, S_2, S_3, \ldots, S_n$ such that S and S_1 share an edge, S_n and T share an edge, and for all positive integers $k < n$, S_k and S_{k+1} share an edge.

We say a polyomino of a given rectangle *spans* the rectangle if for each of the four edges of the rectangle the polyomino contains a square whose edge lies on it.

What is the minimum number of unit squares a polyomino can have if it spans a 128-by-343 rectangle?

17. [±8] Find the number of *pentominoes* (5-square polyominoes) that span a 3-by-3 rectangle, where polyominoes that are flips or rotations of each other are considered the same polyomino.

18. [±5] Call the pentominoes found in the last problem *square pentominoes.* Just like dominos and ominos can be used to tile regions of the plane, so can square pentominoes. In particular, a *square pentomino tiling* of a region of the plane is a way of covering it (and only it) completely by nonoverlapping square pentominoes. How many square pentomino tilings are there of a 12-by-12 rectangle?

19. [**8**] For how many integers a ($1 \leq a \leq 200$) is the number a^a a square?

20. [**7**] The Antarctican language has an alphabet of just 16 letters. Interestingly, every word in the language has exactly 3 letters, and it is known that no word's first letter equals any word's last letter (for instance, if the alphabet were $\{a, b\}$ then aab and aaa could not both be words in the language because a is the first letter of a word and the last letter of a word; in fact, just aaa alone couldn't be in the language). Given this, determine the maximum possible number of words in the language.

21. [**7**] The Dyslexian alphabet consists of consonants and vowels. It so happens that a finite sequence of letters is a word in Dyslexian precisely if it alternates between consonants and vowels (it may begin with either). There are 4800 five-letter words in Dyslexian. How many letters are in the alphabet?

22. [**5**] A *path* of length n is a sequence of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with integer coordinates such that for all i between 1 and $n-1$ inclusive, either

(1) $x_{i+1} = x_i + 1$ and $y_{i+1} = y_i$ (in which case we say the ith step is *rightward*) or

(2) $x_{i+1} = x_i$ and $y_{i+1} = y_i + 1$ (in which case we say that the ith step is *upward*).

This path is said to *start* at (x_1, y_1) and *end* at (x_n, y_n). Let $P(a, b)$, for a and b nonnegative integers, be the number of paths that start at $(0, 0)$ and end at (a, b).

Find $\sum_{i=0}^{10} P(i, 10-i)$.

23. [**5**] Find $P(7, 3)$.

24. [**7**] A *restricted path* of length n is a path of length n such that for all i between 1 and $n-2$ inclusive, if the ith step is upward, the $i+1$st step must be rightward.

Find the number of restricted paths that start at $(0, 0)$ and end at $(7, 3)$.

25. [±4] A math professor stands up in front of a room containing 100 very smart math students and says, "Each of you has to write down an integer between 0 and 100, inclusive, to guess 'two-thirds of the average of all the responses.' Each student who guesses the highest integer that is not higher than two-thirds of the average of all responses will receive a prize." If among all the students it is common knowledge that everyone will write down the best response, and there is no communication between students, what single integer should each of the 100 students write down?

26. [±4] Another professor enters the same room and says, "Each of you has to write down an integer between 0 and 200. I will then compute X, the number that is 3 greater than half the average of all the numbers that you will have written down. Each student who writes down the number closest to X (either above or below X) will receive a prize." One student, who misunderstood the question, announces to the class that he will write the number 107. If among the other 99 students it is common knowledge that all 99 of them will write down the best response, and there is no further communication between students, what single integer should each of the 99 students write down?

27. [7] Consider the two hands of an analog clock, each of which moves with constant angular velocity. Certain positions of these hands are possible (e.g. the hour hand halfway between the 5 and 6 and the minute hand exactly at the 6), while others are impossible (e.g. the hour hand exactly at the 5 and the minute hand exactly at the 6). How many different positions are there that would remain possible if the hour and minute hands were switched?

28. [6] Count how many 8-digit numbers there are that contain exactly four nines as digits.

29. [8] A sequence $s_0, s_1, s_2, s_3, \ldots$ is defined by $s_0 = s_1 = 1$ and, for every positive integer n, $s_{2n} = s_n, s_{4n+1} = s_{2n+1}, s_{4n-1} = s_{2n-1} + s_{2n-1}^2/s_{n-1}$. What is the value of s_{1000}?

30. [9] A conical flask contains some water. When the flask is oriented so that its base is horizontal and lies at the bottom (so that the vertex is at the top), the water is 1 inch deep. When the flask is turned upside-down, so that the vertex is at the bottom, the water is 2 inches deep. What is the height of the cone?

31. [**6**] Express, as concisely as possible, the value of the product

$$(0^3-350)(1^3-349)(2^3-348)(3^3-347)\cdots(349^3-1)(350^3-0).$$

32. [**9**] Two circles have radii 13 and 30, and their centers are 41 units apart. The line through the centers of the two circles intersects the smaller circle at two points; let A be the one outside the larger circle. Suppose B is a point on the smaller circle and C a point on the larger circle such that B is the midpoint of AC. Compute the distance AC.

33. [**8**] The expression $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Find the value of

$$\left\lfloor \frac{2002!}{2001!+2000!+1999!+\cdots+1!} \right\rfloor.$$

34. [**7**] Points P and Q are 3 units apart. A circle centered at P with a radius of $\sqrt{3}$ units intersects a circle centered at Q with a radius of 3 units at points A and B. Find the area of quadrilateral APBQ.

35. [$\pm$7] Suppose a, b, c, d are real numbers such that

$$|a-b|+|c-d|=99; \qquad |a-c|+|b-d|=1.$$

Determine all possible values of $|a-d|+|b-c|$.

36. [$\pm$6] Find the set consisting of all real values of x such that the three numbers $2^x, 2^{x^2}, 2^{x^3}$ form a non-constant arithmetic progression (in that order).

37. [8] Call a positive integer "mild" if its base-3 representation never contains the digit 2. How many values of n ($1 \leq n \leq 1000$) have the property that n and n^2 are both mild?

38. [6] Massachusetts Avenue is ten blocks long. One boy and one girl live on each block. They want to form friendships such that each boy is friends with exactly one girl and vice-versa. Nobody wants a friend living more than one block away (but they may be on the same block). How many pairings are possible?

39. [7] In the x-y plane, draw a circle of radius 2 centered at $(0, 0)$. Color the circle red above the line $y = 1$, color the circle blue below the line $y = -1$, and color the rest of the circle white. Now consider an arbitrary straight line at distance 1 from the circle. We color each point P of the line with the color of the closest point to P on the circle. If we pick such an arbitrary line, randomly oriented, what is the probability that it contains red, white, and blue points?

40. [9] Find the volume of the three-dimensional solid given by the inequality $\sqrt{x^2 + y^2} + |z| \leq 1$.

41. [9] For any integer n, define $\lfloor n \rfloor$ as the greatest integer less than or equal to n. For any positive integer n, let

$$f(n) = \lfloor n \rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor .$$

For how many values of n, $1 \leq n \leq 100$, is $f(n)$ odd?

42. [$\pm$10] Find all the integers $n > 1$ with the following property: the numbers $1, 2, \ldots, n$ can be arranged in a line so that, of any two adjacent numbers, one is divisible by the other.

43. **[9]** Given that a, b, c are positive integers satisfying

$$a + b + c = \gcd(a, b) + \gcd(b, c) + \gcd(c, a) + 120,$$

determine the maximum possible value of a.

44. **[5]** The unknown real numbers x, y, z satisfy the equations

$$\frac{x+y}{1+z} = \frac{1-z+z^2}{x^2-xy+y^2}; \qquad \frac{x-y}{3-z} = \frac{9+3z+z^2}{x^2+xy+y^2}.$$

Find x.

45. **[9]** Find the number of sequences $a_1, a_2, \ldots, a_{10}$ of positive integers with the property that $a_{n+2} = a_{n+1} + a_n$ for $n = 1, 2, \ldots, 8$, and $a_{10} = 2002$.

46. [± 6] Points A, B, C in the plane satisfy $\overline{AB} = 2002, \overline{AC} = 9999$. The circles with diameters AB and AC intersect at A and D. If $\overline{AD} = 37$, what is the shortest distance from point A to line BC?

47. **[9]** The real function f has the property that, whenever a, b, n are positive integers such that $a + b = 2^n$, the equation $f(a) + f(b) = n^2$ holds. What is $f(2002)$?

48. **[9]** A *permutation* of a finite set is a one-to-one function from the set to itself; for instance, one permutation of $\{1, 2, 3, 4\}$ is the function π defined such that $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 4$, and $\pi(4) = 2$. How many permutations π of the set $\{1, 2, \ldots, 10\}$ have the property that $\pi(i) \neq i$ for each $i = 1, 2, \ldots, 10$, but $\pi(\pi(i)) = i$ for each i?

49. **[7]** Two integers are *relatively prime* if they don't share any common factors, i.e. if their greatest common divisor is 1. Define $\varphi(n)$ as the number of positive integers that are less than n and relatively prime to n. Define $\varphi_d(n)$ as the number of positive integers that are less than dn and relatively prime to n.

What is the least n such that $\varphi_x(n) = 64000$, where $x = \varphi_y(n)$, where $y = \varphi(n)$?

50. **[6]** Give the set of all positive integers n such that $\varphi(n) = 2002^2 - 1$.

51. **[10]** Define $\varphi^k(n)$ as the number of positive integers that are less than or equal to n/k and relatively prime to n. Find $\phi^{2001}(2002^2 - 1)$. (Hint: $\phi(2003) = 2002$.)

52. [± 8] Let $ABCD$ be a quadrilateral, and let E, F, G, H be the respective midpoints of AB, BC, CD, DA. If $EG = 12$ and $FH = 15$, what is the maximum possible area of $ABCD$?

53. [**10**] ABC is a triangle with points E, F on sides AC, AB, respectively. Suppose that BE, CF intersect at X. It is given that $AF/FB = (AE/EC)^2$ and that X is the midpoint of BE. Find the ratio CX/XF.

54. [**10**] How many pairs of integers (a, b), with $1 \leq a \leq b \leq 60$, have the property that b is divisible by a and $b + 1$ is divisible by $a + 1$?

55. [**10**] A sequence of positive integers is given by $a_1 = 1$ and $a_n = \gcd(a_{n-1}, n) + 1$ for $n > 1$. Calculate a_{2002}.

56. [± 6] x, y are positive real numbers such that $x + y^2 = xy$. What is the smallest possible value of x?

57. [**9**] How many ways, without taking order into consideration, can 2002 be expressed as the sum of 3 positive integers (for instance, $1000 + 1000 + 2$ and $1000 + 2 + 1000$ are considered to be the same way)?

58. [**8**] A sequence is defined by $a_0 = 1$ and $a_n = 2^{a_{n-1}}$ for $n \geq 1$. What is the last digit (in base 10) of a_{15}?

59. [**7**] Determine the value of

$$1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - 4 \cdot 5 + \cdots + 2001 \cdot 2002.$$

60. [**10**] A 5×5 square grid has the number -3 written in the upper-left square and the number 3 written in the lower-right square. In how many ways can the remaining squares be filled in with integers so that any two adjacent numbers differ by 1, where two squares are adjacent if they share a common edge (but not if they share only a corner)?

61. Bob Barker went back to school for a PhD in math, and decided to raise the intellectual level of *The Price is Right* by having contestants guess how many objects exist of a certain type, without going over. The number of points you will get is the percentage of the correct answer, divided by 10, with no points for going over (i.e. a maximum of 10 points).

Let's see the first object for our contestants...a *table* of shape $(5, 4, 3, 2, 1)$ is an arrangement of the integers 1 through 15 with five numbers in the top row, four in the next, three in the next, two in the next, and one in the last, such that each row and each column is increasing (from left to right, and top to bottom, respectively). For instance:

1	2	3	4	5
6	7	8	9	
10	11	12		
13	14			
15				

is one table. How many tables are there?

62. Our next object up for bid is an arithmetic progression of primes. For example, the primes 3, 5, and 7 form an arithmetic progression of length 3. What is the largest possible length of an arithmetic progression formed of positive primes less than 1,000,000? Be prepared to justify your answer.

63. Our third and final item comes to us from Germany, I mean Geometry. It is known that a regular n-gon can be constructed with straightedge and compass if n is a prime that is 1 plus a power of 2. It is also possible to construct a $2n$-gon whenever an n-gon is constructible, or a $p_1 p_2 \cdots p_m$-gon where the p_i's are distinct primes of the above form. What is really interesting is that these conditions, together with the fact that we can construct a square, is that they give us all constructible regular n-gons. What is the largest n less than 4,300,000,000 such that a regular n-gon is constructible?

Help control the pet population. Have your pets spayed or neutered. Bye-bye.