Team Event
HMMT 2002

Palindromes. A *palindrome* is a positive integer n not divisible by 10 such that if you write the decimal digits of n in reverse order, the number you get is n itself. For instance, the numbers 4 and 25752 are palindromes.

1. [15] Determine the number of palindromes that are less than 1000.

2. [30] Determine the number of four-digit integers n such that n and $2n$ are both palindromes.

3. [40] Suppose that a positive integer n has the property that n, $2n$, $3n$, ..., $9n$ are all palindromes. Prove that the decimal digits of n are all zeros or ones.

Floor functions. The notation $\lfloor x \rfloor$ stands for the largest integer less than or equal to x.

4. [15] Let n be an integer. Prove that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor = n.$$

5. [20] Prove for integers n that

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

In problems 6–7 you may use without proof the known summations

$$\sum_{n=1}^{L} n = n(n+1)/2 \quad \text{and} \quad \sum_{n=1}^{L} n^3 = n^2(n+1)^2/4 \quad \text{for positive integers } L.$$

6. [20] For positive integers L, let $S_L = \sum_{n=1}^{L} \lfloor n/2 \rfloor$. Determine all L for which S_L is a square number.

7. [45] Let $T_L = \sum_{n=1}^{L} \lfloor n^3/9 \rfloor$ for positive integers L. Determine all L for which T_L is a square number.

Luck of the dice. Problems 8–12 concern a two-player game played on a board consisting of fourteen spaces in a row. The leftmost space is labeled *START*, and the rightmost space is labeled *END*. Each of the twelve other squares, which we number 1 through 12 from left to right, may be blank or may be labeled with an arrow pointing to the right. The term *blank square* will refer to one of these twelve squares that is not labeled with an arrow. The set of blank squares on the board will be called a *board configuration*; the board below uses the configuration $\{1, 2, 3, 4, 7, 8, 10, 11, 12\}$.

START					$\Rightarrow$	$\Rightarrow$			$\Rightarrow$				*END*
	1	2	3	4	5	6	7	8	9	10	11	12	

For $i \in \{1, 2\}$, player i has a die that produces each integer from 1 to s_i with probability $1/s_i$. Here s_1 and s_2 are positive integers fixed before the game begins. The game rules are as follows:

1. The players take turns alternately, and player 1 takes the first turn.
2. On each of his turns, player i rolls his die and moves his piece to the right by the number of squares that he rolled. If his move ends on a square marked with an arrow, he moves his piece forward another s_i squares. If that move ends on an arrow, he moves another s_i squares, repeating until his piece comes to rest on a square without an arrow.
3. If a player's move would take him past the *END* square, instead he lands on the *END* square.
4. Whichever player reaches the *END* square first wins.

As an example, suppose that $s_1 = 3$ and the first player is on square 4 in the sample board shown above. If the first player rolls a 2, he moves to square 6, then to square 9, finally coming to rest on square 12. If the second player does not reach the *END* square on her next turn, the first player will necessarily win on his next turn, as he must roll at least a 1.

8. [35] In this problem only, assume that $s_1 = 4$ and that exactly one board square, say square number n, is marked with an arrow. Determine all choices of n that maximize the average distance in squares the first player will travel in his first two turns.

9. [30] In this problem suppose that $s_1 = s_2$. Prove that for each board configuration, the first player wins with probability strictly greater than $\frac{1}{2}$.

10. [30] Exhibit a configuration of the board and a choice of s_1 and s_2 so that $s_1 > s_2$, yet the *second* player wins with probability strictly greater than $\frac{1}{2}$.

11. [55] In this problem assume $s_1 = 3$ and $s_2 = 2$. Determine, with proof, the nonnegative integer k with the following property:

1. For every board configuration with strictly fewer than k blank squares, the first player wins with probability strictly greater than $\frac{1}{2}$; but
2. there exists a board configuration with exactly k blank squares for which the second player wins with probability strictly greater than $\frac{1}{2}$.

12. [65] Now suppose that before the game begins, the players choose the initial game state as follows:

1. The first player chooses s_1 subject to the constraint that $2 \leq s_1 \leq 5$; then
2. the second player chooses s_2 subject to the constraint that $2 \leq s_2 \leq 5$ and then specifies the board configuration.

Prove that the second player can always make her decisions so that she will win the game with probability strictly greater than $\frac{1}{2}$.