Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Algebra Subject Test

1. Find the smallest value of x such that $a \geq 14\sqrt{a} - x$ for all nonnegative a.

2. Compute $\frac{\tan^2(20^\circ) - \sin^2(20^\circ)}{\tan^2(20^\circ)\sin^2(20^\circ)}$.

3. Find the smallest n such that $n!$ ends in 290 zeroes.

4. Simplify: $2\sqrt{1.5 + \sqrt{2}} - (1.5 + \sqrt{2})$.

5. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that n_1 of the given numbers are equal to 1, n_2 of them are equal to 2, $\ldots$, n_{2003} of them are equal to 2003. Find the largest possible value of

$$n_2 + 2n_3 + 3n_4 + \cdots + 2002n_{2003}.$$

6. Let $a_1 = 1$, and let $a_n = \lfloor n^3/a_{n-1} \rfloor$ for $n > 1$. Determine the value of a_{999}.

7. Let a, b, c be the three roots of $p(x) = x^3 + x^2 - 333x - 1001$. Find $a^3 + b^3 + c^3$.

8. Find the value of $\frac{1}{3^2+1} + \frac{1}{4^2+2} + \frac{1}{5^2+3} + \cdots$.

9. For how many integers n, for $1 \leq n \leq 1000$, is the number $\frac{1}{2}\binom{2n}{n}$ even?

10. Suppose $P(x)$ is a polynomial such that $P(1) = 1$ and

$$\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$$

for all real x for which both sides are defined. Find $P(-1)$.