Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Calculus Subject Test

1. A point is chosen randomly with uniform distribution in the interior of a circle of radius 1. What is its expected distance from the center of the circle?

2. A particle moves along the x-axis in such a way that its velocity at position x is given by the formula $v(x) = 2 + \sin x$. What is its acceleration at $x = \frac{\pi}{6}$?

3. What is the area of the region bounded by the curves $y = x^{2003}$ and $y = x^{1/2003}$ and lying above the x-axis?

4. The sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies $\lim_{n\to\infty}(x_{2n} + x_{2n+1}) = 315$ and $\lim_{n\to\infty}(x_{2n} + x_{2n-1}) = 2003$. Evaluate $\lim_{n\to\infty}(x_{2n}/x_{2n+1})$.

5. Find the minimum distance from the point $(0, 5/2)$ to the graph of $y = x^4/8$.

6. For n an integer, evaluate
$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2-0^2}} + \frac{1}{\sqrt{n^2-1^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right).$$

7. For what value of $a > 1$ is
$$\int_a^{a^2} \frac{1}{x} \log \frac{x-1}{32}\, dx$$
minimum?

8. A right circular cone with a height of 12 inches and a base radius of 3 inches is filled with water and held with its vertex pointing downward. Water flows out through a hole at the vertex at a rate in cubic inches per second numerically equal to the height of the water in the cone. (For example, when the height of the water in the cone is 4 inches, water flows out at a rate of 4 cubic inches per second.) Determine how many seconds it will take for all of the water to flow out of the cone.

9. Two differentiable real functions $f(x)$ and $g(x)$ satisfy
$$\frac{f'(x)}{g'(x)} = e^{f(x)-g(x)}$$
for all x, and $f(0) = g(2003) = 1$. Find the largest constant c such that $f(2003) > c$ for all such functions f, g.

10. Evaluate
$$\int_{-\infty}^{\infty} \frac{1-x^2}{1+x^4}\, dx.$$