Harvard-MIT Mathematics Tournament

March 15, 2003

Team Round

Completions and Configurations

Given a set A and a nonnegative integer k, the *k-completion* of A is the collection of all k-element subsets of A, and a *k-configuration* of A is any subset of the k-completion of A (including the empty set and the entire k-completion). For instance, the 2-completion of $A = \{1, 2, 3\}$ is $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and the 2-configurations of A are

$$\begin{array}{cc} \{\} & \{\{1,2\}\} \\ \{\{1,3\}\} & \{\{2,3\}\} \\ \{\{1,2\},\{1,3\}\} & \{\{1,2\},\{2,3\}\} \\ \{\{1,3\},\{2,3\}\} & \{\{1,2\},\{1,3\},\{2,3\}\} \end{array}$$

The *order* of an element a of A with respect to a given k-configuration of A is the number of subsets in the k-configuration that contain a. A k-configuration of a set A is *consistent* if the order of every element of A is the same, and the *order* of a consistent k-configuration is this common value.

1. (a) **[10]** How many k-configurations are there of a set that has n elements?

   (b) **[10]** How many k-configurations that have m elements are there of a set that has n elements?

2. **[15]** Suppose A is a set with n elements, and k is a divisor of n. Find the number of consistent k-configurations of A of order 1.

3. (a) **[15]** Let $A_n = \{a_1, a_2, a_3, \dots, a_n, b\}$, for $n \geq 3$, and let C_n be the 2-configuration consisting of $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq n-1$, $\{a_1, a_n\}$, and $\{a_i, b\}$ for $1 \leq i \leq n$. Let $S_e(n)$ be the number of subsets of C_n that are consistent of order e. Find $S_e(101)$ for $e = 1, 2$, and 3.

   (b) **[20]** Let $A = \{V, W, X, Y, Z, v, w, x, y, z\}$. Find the number of subsets of the 2-configuration

   $$\{\ \{V,W\},\{W,X\},\{X,Y\},\{Y,Z\},\{Z,V\},\ \{v,x\},\{v,y\},\{w,y\},\{w,z\},\{x,z\},$$
   $$\{V,v\},\{W,w\},\{X,x\},\{Y,y\},\{Z,z\}\ \}$$

   that are consistent of order 1.

   (c) **[30]** Let $A = \{a_1, b_1, a_2, b_2, \dots, a_{10}, b_{10}\}$, and consider the 2-configuration C consisting of $\{a_i, b_i\}$ for all $1 \leq i \leq 10$, $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq 9$, and $\{b_i, b_{i+1}\}$ for all $1 \leq i \leq 9$. Find the number of subsets of C that are consistent of order 1.

Define a k-configuration of A to be *m-separable* if we can label each element of A with an integer from 1 to m (inclusive) so that there is no element E of the k-configuration all of whose elements are assigned the same integer. If C is any subset of A, then C is *m-separable* if we can assign an integer from 1 to m to each element of C so that there is no element E of the k-configuration such that $E \subseteq C$ and all elements of E are assigned the same integer.

4. (a) **[15]** Suppose A has n elements, where $n \geq 2$, and C is a 2-configuration of A that is not m-separable for any $m < n$. What is (in terms of n) the smallest number of elements that C can have?

   (b) **[15]** Show that every 3-configuration of an n-element set A is m-separable for every integer $m \geq n/2$.

   (c) **[25]** Fix $k \geq 2$, and suppose A has k^2 elements. Show that any k-configuration of A with fewer than $\binom{k^2-1}{k-1}$ elements is k-separable.

5. **[30]** Let $B_k(n)$ be the largest number of elements in a 2-separable k-configuration of a set with $2n$ elements ($2 \leq k \leq n$). Find a closed-form expression (i.e. an expression not involving any sums or products with an variable number of terms) for $B_k(n)$.

6. **[40]** Prove that any 2-configuration containing e elements is m-separable for some $m \leq \frac{1}{2} + \sqrt{2e + \frac{1}{4}}$.

A *cell* of a 2-configuration of a set A is a nonempty subset C of A such that

i. for any two distinct elements a, b of C, there exists a sequence $c_0, c_1, \ldots, c_n$ of elements of A with $c_0 = a, c_n = b$, and such that $\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}$ are all elements of the 2-configuration, and

ii. if a is an element of C and b is an element of A but not of C, there does NOT exist a sequence $c_0, c_1, \ldots, c_n$ of elements of A with $c_0 = a, c_n = b$, and such that $\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}$ are all elements of the 2-configuration.

Also, we define a 2-configuration of A to be *barren* if there is no subset $\{a_0, a_1, \ldots, a_n\}$ of A, with $n \geq 2$, such that $\{a_0, a_1\}, \{a_1, a_2\}, \ldots, \{a_{n-1}, a_n\}$ and $\{a_n, a_0\}$ are all elements of the 2-configuration.

7. **[20]** Show that, given any 2-configuration of a set A, every element of A belongs to exactly one cell.

8. (a) **[15]** Given a set A with $n \geq 1$ elements, find the number of consistent 2-configurations of A of order 1 with exactly 1 cell.

   (b) **[25]** Given a set A with 10 elements, find the number of consistent 2-configurations of A of order 2 with exactly 1 cell.

   (c) **[25]** Given a set A with 10 elements, find the number of consistent 2-configurations of order 2 with exactly 2 cells.

9. (a) **[15]** Show that if every cell of a 2-configuration of a finite set A is m-separable, then the whole 2-configuration is m-separable.

   (b) **[30]** Show that any barren 2-configuration of a finite set A is 2-separable.

10. **[45]** Show that every consistent 2-configuration of order 4 on a finite set A has a subset that is a consistent 2-configuration of order 2.