Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Algebra Subject Test

1. How many ordered pairs of integers (a,b) satisfy all of the following inequalities?

$$\begin{aligned} a^2+b^2 &< 16 \\ a^2+b^2 &< 8a \\ a^2+b^2 &< 8b \end{aligned}$$

2. Find the largest number n such that (2004!)! is divisible by $((n!)!)!$.

3. Compute:

$$\left\lfloor \frac{2005^3}{2003 \cdot 2004} - \frac{2003^3}{2004 \cdot 2005} \right\rfloor .$$

4. Evaluate the sum

$$\frac{1}{2\lfloor\sqrt{1}\rfloor+1} + \frac{1}{2\lfloor\sqrt{2}\rfloor+1} + \frac{1}{2\lfloor\sqrt{3}\rfloor+1} + \cdots + \frac{1}{2\lfloor\sqrt{100}\rfloor+1}.$$

5. There exists a positive real number x such that $\cos(\tan^{-1}(x)) = x$. Find the value of x^2.

6. Find all real solutions to $x^4 + (2-x)^4 = 34$.

7. If x, y, k are positive reals such that

$$3 = k^2\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) + k\left(\frac{x}{y} + \frac{y}{x}\right),$$

find the maximum possible value of k.

8. Let x be a real number such that $x^3 + 4x = 8$. Determine the value of $x^7 + 64x^2$.

9. A sequence of positive integers is defined by $a_0 = 1$ and $a_{n+1} = a_n^2 + 1$ for each $n \geq 0$. Find $\gcd(a_{999}, a_{2004})$.

10. There exists a polynomial P of degree 5 with the following property: if z is a complex number such that $z^5 + 2004z = 1$, then $P(z^2) = 0$. Calculate the quotient $P(1)/P(-1)$.