Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Calculus Subject Test

1. Let $f(x) = \sin(\sin x)$. Evaluate $\lim_{h\to 0} \frac{f(x+h)-f(h)}{x}$ at $x = \pi$.

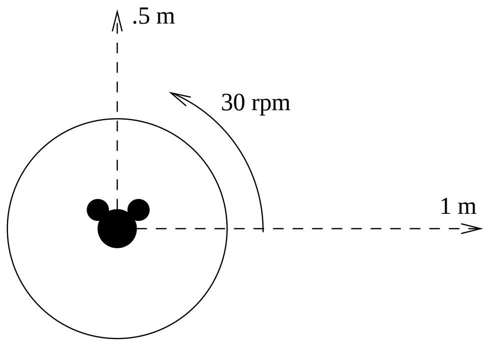
2. Suppose the function $f(x) - f(2x)$ has derivative 5 at $x = 1$ and derivative 7 at $x = 2$. Find the derivative of $f(x) - f(4x)$ at $x = 1$.

3. Find $\lim_{x\to\infty}(\sqrt[3]{x^3+x^2} - \sqrt[3]{x^3-x^2})$.

4. Let $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos x)))))))$, and suppose that the number a satisfies the equation $a = \cos a$. Express $f'(a)$ as a polynomial in a.

5. A mouse is sitting in a toy car on a negligibly small turntable. The car cannot turn on its own, but the mouse can control when the car is launched and when the car stops (the car has brakes). When the mouse chooses to launch, the car will immediately leave the turntable on a straight trajectory at 1 meter per second.

   Suddenly someone turns on the turntable; it spins at 30 rpm. Consider the set S of points the mouse can reach in his car within 1 second after the turntable is set in motion. (For example, the arrows in the figure below represent two possible paths the mouse can take.) What is the area of S, in square meters?



6. For $x > 0$, let $f(x) = x^x$. Find all values of x for which $f(x) = f'(x)$.

7. Find the area of the region in the xy-plane satisfying $x^6 - x^2 + y^2 \leq 0$.

8. If x and y are real numbers with $(x+y)^4 = x - y$, what is the maximum possible value of y?

9. Find the positive constant c_0 such that the series

$$\sum_{n=0}^{\infty} \frac{n!}{(cn)^n}$$

   converges for $c > c_0$ and diverges for $0 < c < c_0$.

10. Let $P(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$. Let $P^{[1]}(x) = P(x)$, and for $n \geq 1$, let $P^{[n+1]}(x) = P^{[n]}(P(x))$. Evaluate $\int_0^1 P^{[2004]}(x)\,dx$.