Harvard-MIT Mathematics Tournament
February 28, 2004

Individual Round: General Test, Part 1

1. There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different rooms will have people in them?

2. What is the largest whole number that is equal to the product of its digits?

3. Suppose f is a function that assigns to each real number x a value $f(x)$, and suppose the equation

   $$f(x_1 + x_2 + x_3 + x_4 + x_5) = f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) - 8$$

   holds for all real numbers x_1, x_2, x_3, x_4, x_5. What is $f(0)$?

4. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

5. A rectangle has perimeter 10 and diagonal $\sqrt{15}$. What is its area?

6. Find the ordered quadruple of digits (A, B, C, D), with $A > B > C > D$, such that

$$
\begin{array}{r}
ABCD \\
- \quad \underline{DCBA} \\
= \quad BDAC.
\end{array}
$$

7. Let ACE be a triangle with a point B on segment AC and a point D on segment CE such that BD is parallel to AE. A point Y is chosen on segment AE, and segment CY is drawn. Let X be the intersection of CY and BD. If $CX = 5$, $XY = 3$, what is the ratio of the area of trapezoid $ABDE$ to the area of triangle BCD?
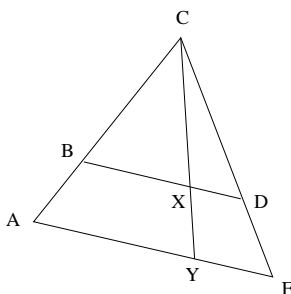
8. You have a 10×10 grid of squares. You write a number in each square as follows: you write $1, 2, 3, \ldots, 10$ from left to right across the top row, then $11, 12, \ldots, 20$ across the second row, and so on, ending with a 100 in the bottom right square. You then write a second number in each square, writing $1, 2, \ldots, 10$ in the first column (from top to bottom), then $11, 12, \ldots, 20$ in the second column, and so forth.

   When this process is finished, how many squares will have the property that their two numbers sum to 101?

9. Urn A contains 4 white balls and 2 red balls. Urn B contains 3 red balls and 3 black balls. An urn is randomly selected, and then a ball inside of that urn is removed. We then repeat the process of selecting an urn and drawing out a ball, without returning the first ball. What is the probability that the first ball drawn was red, given that the second ball drawn was black?

10. A floor is tiled with equilateral triangles of side length 1, as shown. If you drop a needle of length 2 somewhere on the floor, what is the largest number of triangles it could end up intersecting? (Only count the triangles whose interiors are met by the needle — touching along edges or at corners doesn't qualify.)