Harvard-MIT Mathematics Tournament

February 28, 2004

Guts Round

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1. **[5]** Find the value of

$$\binom{6}{1}2^1 + \binom{6}{2}2^2 + \binom{6}{3}2^3 + \binom{6}{4}2^4 + \binom{6}{5}2^5 + \binom{6}{6}2^6.$$

2. **[5]** If the three points

$$(1, a, b)$$
$$(a, 2, b)$$
$$(a, b, 3)$$

are collinear (in 3-space), what is the value of $a + b$?

3. **[5]** If the system of equations

$$\begin{aligned} |x + y| &= 99 \\ |x - y| &= c \end{aligned}$$

has exactly two real solutions (x, y), find the value of c.

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4. **[6]** A tree grows in a rather peculiar manner. Lateral cross-sections of the trunk, leaves, branches, twigs, and so forth are circles. The trunk is 1 meter in diameter to a height of 1 meter, at which point it splits into two sections, each with diameter .5 meter. These sections are each one meter long, at which point they each split into two sections, each with diameter .25 meter. This continues indefinitely: every section of tree is 1 meter long and splits into two smaller sections, each with half the diameter of the previous.

   What is the total volume of the tree?

5. **[6]** Augustin has six $1 \times 2 \times \pi$ bricks. He stacks them, one on top of another, to form a tower six bricks high. Each brick can be in any orientation so long as it rests flat on top of the next brick below it (or on the floor). How many distinct heights of towers can he make?

6. **[6]** Find the smallest integer n such that $\sqrt{n + 99} - \sqrt{n} < 1$.

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7. **[6]** Find the shortest distance from the line $3x+4y=25$ to the circle $x^2+y^2=6x-8y$.

8. **[6]** I have chosen five of the numbers $\{1,2,3,4,5,6,7\}$. If I told you what their product was, that would not be enough information for you to figure out whether their sum was even or odd. What is their product?

9. **[6]** A positive integer n is *picante* if $n!$ ends in the same number of zeroes whether written in base 7 or in base 8. How many of the numbers $1,2,\ldots,2004$ are picante?

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10. **[7]** Let $f(x)=x^2+x^4+x^6+x^8+\cdots$, for all real x such that the sum converges. For how many real numbers x does $f(x)=x$?

11. **[7]** Find all numbers n with the following property: there is exactly one set of 8 different positive integers whose sum is n.

12. **[7]** A convex quadrilateral is drawn in the coordinate plane such that each of its vertices (x,y) satisfies the equations $x^2+y^2=73$ and $xy=24$. What is the area of this quadrilateral?

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13. **[7]** Find all positive integer solutions (m,n) to the following equation:

$$m^2 = 1! + 2! + \cdots + n!.$$

14. **[7]** If $a_1=1, a_2=0$, and $a_{n+1}=a_n+\frac{a_{n+2}}{2}$ for all $n\geq 1$, compute a_{2004}.

15. **[7]** A regular decagon $A_0A_1A_2\cdots A_9$ is given in the plane. Compute $\angle A_0A_3A_7$ in degrees.

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16. [**8**] An n-string is a string of digits formed by writing the numbers $1, 2, \ldots, n$ in some order (in base ten). For example, one possible 10-string is

$$35728910461$$

What is the smallest $n > 1$ such that there exists a palindromic n-string?

17. [**8**] Kate has four red socks and four blue socks. If she randomly divides these eight socks into four pairs, what is the probability that none of the pairs will be mismatched? That is, what is the probability that each pair will consist either of two red socks or of two blue socks?

18. [**8**] On a spherical planet with diameter $10,000$ km, powerful explosives are placed at the north and south poles. The explosives are designed to vaporize all matter within $5,000$ km of ground zero and leave anything beyond $5,000$ km untouched. After the explosives are set off, what is the new surface area of the planet, in square kilometers?

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19. [**8**] The Fibonacci numbers are defined by $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. If the number

$$\frac{F_{2003}}{F_{2002}} - \frac{F_{2004}}{F_{2003}}$$

is written as a fraction in lowest terms, what is the numerator?

20. [**8**] Two positive rational numbers x and y, when written in lowest terms, have the property that the sum of their numerators is 9 and the sum of their denominators is 10. What is the largest possible value of $x + y$?

21. [**8**] Find all ordered pairs of integers (x, y) such that $3^x 4^y = 2^{x+y} + 2^{2(x+y)-1}$.

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22. [**9**] I have written a strictly increasing sequence of six positive integers, such that each number (besides the first) is a multiple of the one before it, and the sum of all six numbers is 79. What is the largest number in my sequence?

23. [**9**] Find the largest integer n such that $3^{512} - 1$ is divisible by 2^n.

24. [**9**] We say a point is *contained* in a square if it is in its interior or on its boundary. Three unit squares are given in the plane such that there is a point contained in all three. Furthermore, three points A, B, C, are given, each contained in at least one of the squares. Find the maximum area of triangle ABC.

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25. [**9**] Suppose $x^3 - ax^2 + bx - 48$ is a polynomial with three positive roots p, q, and r such that $p < q < r$. What is the minimum possible value of $1/p + 2/q + 3/r$?

26. [**9**] How many of the integers $1, 2, \ldots, 2004$ can be represented as $(mn + 1)/(m + n)$ for positive integers m and n?

27. [**9**] A regular hexagon has one side along the diameter of a semicircle, and the two opposite vertices on the semicircle. Find the area of the hexagon if the diameter of the semicircle is 1.

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28. [**10**] Find the value of

$$\binom{2003}{1} + \binom{2003}{4} + \binom{2003}{7} + \cdots + \binom{2003}{2002}.$$

29. [**10**] A regular dodecahedron is projected orthogonally onto a plane, and its image is an n-sided polygon. What is the smallest possible value of n?

30. [**10**] We have an n-gon, and each of its vertices is labeled with a number from the set $\{1, \ldots, 10\}$. We know that for any pair of distinct numbers from this set there is at least one side of the polygon whose endpoints have these two numbers. Find the smallest possible value of n.

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31. **[10]** P is a point inside triangle ABC, and lines AP, BP, CP intersect the opposite sides BC, CA, AB in points D, E, F, respectively. It is given that $\angle APB = 90°$, and that $AC = BC$ and $AB = BD$. We also know that $BF = 1$, and that $BC = 999$. Find AF.

32. **[10]** Define the sequence $b_0, b_1, \ldots, b_{59}$ by

$$b_i = \begin{cases} 1 & \text{if i is a multiple of 3} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{a_i\}$ be a sequence of elements of $\{0, 1\}$ such that

$$b_n \equiv a_{n-1} + a_n + a_{n+1} \pmod 2$$

for $0 \leq n \leq 59$ ($a_0 = a_{60}$ and $a_{-1} = a_{59}$). Find all possible values of $4a_0 + 2a_1 + a_2$.

33. **[10]** A plane P slices through a cube of volume 1 with a cross-section in the shape of a regular hexagon. This cube also has an inscribed sphere, whose intersection with P is a circle. What is the area of the region inside the regular hexagon but outside the circle?

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34. **[12]** Find the number of 20-tuples of integers $x_1, \ldots, x_{10}, y_1, \ldots, y_{10}$ with the following properties:

- $1 \leq x_i \leq 10$ and $1 \leq y_i \leq 10$ for each i;
- $x_i \leq x_{i+1}$ for $i = 1, \ldots, 9$;
- if $x_i = x_{i+1}$, then $y_i \leq y_{i+1}$.

35. **[12]** There are eleven positive integers n such that there exists a convex polygon with n sides whose angles, in degrees, are unequal integers that are in arithmetic progression. Find the sum of these values of n.

36. **[12]** For a string of P's and Q's, the *value* is defined to be the product of the positions of the P's. For example, the string $PPQPQQ$ has value $1 \cdot 2 \cdot 4 = 8$.

Also, a string is called *antipalindromic* if writing it backwards, then turning all the P's into Q's and vice versa, produces the original string. For example, $PPQPQQ$ is antipalindromic.

There are 2^{1002} antipalindromic strings of length 2004. Find the sum of the reciprocals of their values.

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37. **[15]** Simplify $\prod_{k=1}^{2004} \sin(2\pi k/4009)$.

38. **[15]** Let $S = \{p_1 p_2 \cdots p_n \mid p_1, p_2, \ldots, p_n$ are distinct primes and $p_1, \ldots, p_n < 30\}$. Assume 1 is in S. Let a_1 be an element of S. We define, for all positive integers n:

$$a_{n+1} = a_n/(n+1) \quad \text{if } a_n \text{ is divisible by } n+1;$$

$$a_{n+1} = (n+2)a_n \quad \text{if } a_n \text{ is not divisible by } n+1.$$

How many distinct possible values of a_1 are there such that $a_j = a_1$ for infinitely many j's?

39. **[15]** You want to arrange the numbers $1, 2, 3, \ldots, 25$ in a sequence with the following property: if n is divisible by m, then the nth number is divisible by the mth number. How many such sequences are there?

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40. **[18]** You would like to provide airline service to the 10 cities in the nation of Schizophrenia, by instituting a certain number of two-way routes between cities. Unfortunately, the government is about to divide Schizophrenia into two warring countries of five cities each, and you don't know which cities will be in each new country. All airplane service between the two new countries will be discontinued. However, you want to make sure that you set up your routes so that, for any two cities in the same new country, it will be possible to get from one city to the other (without leaving the country).

    What is the minimum number of routes you must set up to be assured of doing this, no matter how the government divides up the country?

41. **[18]** A tetrahedron has all its faces triangles with sides $13, 14, 15$. What is its volume?

42. **[18]** S is a set of complex numbers such that if $u, v \in S$, then $uv \in S$ and $u^2 + v^2 \in S$. Suppose that the number N of elements of S with absolute value at most 1 is finite. What is the largest possible value of N?

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43. Write down an integer from 0 to 20 inclusive. This problem will be scored as follows: if N is the second-largest number from among the responses submitted, then each team that submits N gets N points, and everyone else gets zero. (If every team picks the same number then nobody gets any points.)

44. Shown on your answer sheet is a 20×20 grid. Place as many queens as you can so that each of them attacks at most one other queen. (A queen is a chess piece that can move any number of squares horizontally, vertically, or diagonally.) It's not very hard to get 20 queens, so you get no points for that, but you get 5 points for each further queen beyond 20. You can mark the grid by placing a dot in each square that contains a queen.

45. A *binary string of length n* is a sequence of n digits, each of which is 0 or 1. The *distance* between two binary strings of the same length is the number of positions in which they disagree; for example, the distance between the strings 01101011 and 00101110 is 3 since they differ in the second, sixth, and eighth positions.

    Find as many binary strings of length 8 as you can, such that the distance between any two of them is at least 3. You get one point per string.