Harvard-MIT Mathematics Tournament

February 28, 2004

Team Round

A Build-It-Yourself Table [**150 points**]

An infinite table of nonnegative integers is constructed as follows: in the top row, some number is 1 and all other numbers are 0's; in each subsequent row, every number is the sum of some two of the three closest numbers in the preceding row. An example of such a table is shown below.

$$\begin{array}{ccccccccccc}
\cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 1 & 3 & 3 & 2 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 2 & 4 & 4 & 6 & 3 & 2 & 0 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The top row (with the one 1) is called row 0; the next row is row 1; the next row is row 2, and so forth.

Note that the following problems require you to prove the statements for *every* table that can be constructed by the process described above, not just for the example shown.

1. [**10**] Show that any number in row n (for $n > 0$) is at most 2^{n-1}.

2. [**20**] What is the earliest row in which the number 2004 may appear?

3. [**35**] Let
$$S(n,r) = \binom{n-1}{r-1} + \binom{n-1}{r} + \binom{n-1}{r+1} + \cdots + \binom{n-1}{n-1}$$
for all $n, r > 0$, and in particular $S(n,r) = 0$ if $r > n > 0$. Prove that the number in row n of the table, r columns to the left of the 1 in the top row, is at most $S(n,r)$. (**Hint**: First prove that $S(n-1,r-1) + S(n-1,r) = S(n,r)$.)

4. [**25**] Show that the sum of all the numbers in row n is at most $(n+2)2^{n-1}$.

A pair of successive numbers in the same row is called a *switch pair* if one number in the pair is even and the other is odd.

5. [**15**] Prove that the number of switch pairs in row n is at most twice the number of odd numbers in row n.

6. [**20**] Prove that the number of odd numbers in row n is at most twice the number of switch pairs in row $n-1$.

7. [**25**] Prove that the number of switch pairs in row n is at most twice the number of switch pairs in row $n-1$.

Written In The Stars [**125 points**]

Suppose S is a finite set with a binary operation $\star$ — that is, for any elements a, b of S, there is defined an element $a \star b$ of S. It is given that $(a \star b) \star (a \star b) = b \star a$ for all $a, b \in S$.

8. [**20**] Prove that $a \star b = b \star a$ for all $a, b \in S$.

Let T be the set of elements of the form $a \star a$ for $a \in S$.

9. [**15**] If b is any element of T, prove that $b \star b = b$.

Now suppose further that $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in S$. (Thus we can write an expression like $a \star b \star c \star d$ without ambiguity.)

10. [**25**] Let a be an element of T. Let the *image* of a be the set of all elements of T that can be represented as $a \star b$ for some $b \in T$. Prove that if c is in the image of a, then $a \star c = c$.
11. [**40**] Prove that there exists an element $a \in T$ such that the equation $a \star b = a$ holds for all $b \in T$.
12. [**25**] Prove that there exists an element $a \in S$ such that the equation $a \star b = a$ holds for all $b \in S$.

Sigma City [**125 points**]

13. [**25**] Let n be a positive odd integer. Prove that

$$\lfloor \log_2 n \rfloor + \lfloor \log_2(n/3) \rfloor + \lfloor \log_2(n/5) \rfloor + \lfloor \log_2(n/7) \rfloor + \cdots + \lfloor \log_2(n/n) \rfloor = (n-1)/2.$$

Let $\sigma(n)$ denote the sum of the (positive) divisors of n, including 1 and n itself.

14. [**30**] Prove that

$$\sigma(1) + \sigma(2) + \sigma(3) + \cdots + \sigma(n) \leq n^2$$

for every positive integer n.

15. [**30**] Prove that

$$\frac{\sigma(1)}{1} + \frac{\sigma(2)}{2} + \frac{\sigma(3)}{3} + \cdots + \frac{\sigma(n)}{n} \leq 2n$$

for every positive integer n.

16. [**40**] Now suppose again that n is odd. Prove that

$$\sigma(1)\lfloor \log_2 n \rfloor + \sigma(3)\lfloor \log_2(n/3) \rfloor + \sigma(5)\lfloor \log_2(n/5) \rfloor + \cdots + \sigma(n)\lfloor \log_2(n/n) \rfloor < n^2/8.$$