Harvard-MIT Mathematics Tournament
February 19, 2005

Individual Round: General Test, Part 1

1. How many real numbers x are solutions to the following equation?

$$|x - 1| = |x - 2| + |x - 3|$$

2. A true-false test has ten questions. If you answer five questions "true" and five "false," your score is guaranteed to be at least four. How many answer keys are there for which this is true?

3. Let $ABCD$ be a regular tetrahedron with side length 2. The plane parallel to edges AB and CD and lying halfway between them cuts $ABCD$ into two pieces. Find the surface area of one of these pieces.

4. Find all real solutions to $x^3 + (x + 1)^3 + (x + 2)^3 = (x + 3)^3$.

5. In how many ways can 4 purple balls and 4 green balls be placed into a 4×4 grid such that every row and column contains one purple ball and one green ball? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

6. In an election, there are two candidates, A and B, who each have 5 supporters. Each supporter, independent of other supporters, has a $\frac{1}{2}$ probability of voting for his or her candidate and a $\frac{1}{2}$ probability of being lazy and not voting. What is the probability of a tie (which includes the case in which no one votes)?

7. If $a, b, c > 0$, what is the smallest possible value of $\left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor$? (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

8. Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors. What is the sum of the ten numbers?

9. A triangular piece of paper of area 1 is folded along a line parallel to one of the sides and pressed flat. What is the minimum possible area of the resulting figure?

10. What is the smallest integer x larger than 1 such that x^2 ends in the same three digits as x does?