Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: General Test, Part 2

1. The volume of a cube (in cubic inches) plus three times the total length of its edges (in inches) is equal to twice its surface area (in square inches). How many inches long is its long diagonal?

2. Find three real numbers $a < b < c$ satisfying:

$$\begin{aligned} a + b + c &= 21/4 \\ 1/a + 1/b + 1/c &= 21/4 \\ abc &= 1. \end{aligned}$$

3. Working together, Jack and Jill can paint a house in 3 days; Jill and Joe can paint the same house in 4 days; or Joe and Jack can paint the house in 6 days. If Jill, Joe, and Jack all work together, how many days will it take them?

4. In how many ways can 8 people be arranged in a line if Alice and Bob must be next to each other, and Carol must be somewhere behind Dan?

5. You and I play the following game on an 8×8 square grid of boxes: Initially, every box is empty. On your turn, you choose an empty box and draw an X in it; if any of the four adjacent boxes are empty, you mark them with an X as well. (Two boxes are adjacent if they share an edge.) We alternate turns, with you moving first, and whoever draws the last X wins. How many choices do you have for a first move that will enable you to guarantee a win no matter how I play?

6. A cube with side length 2 is inscribed in a sphere. A second cube, with faces parallel to the first, is inscribed between the sphere and one face of the first cube. What is the length of a side of the smaller cube?

7. Three distinct lines are drawn in the plane. Suppose there exist exactly n circles in the plane tangent to all three lines. Find all possible values of n.

8. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

9. In how many ways can the cells of a 4×4 table be filled in with the digits $1, 2, \ldots, 9$ so that each of the 4-digit numbers formed by the columns is divisible by each of the 4-digit numbers formed by the rows?

10. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many positive integers less than 2005 can be expressed in the form $\lfloor x \lfloor x \rfloor \rfloor$ for some positive real x?