IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Individual Round: Calculus Test

1. A nonzero polynomial $f(x)$ with real coefficients has the property that $f(x) = f'(x)f''(x)$. What is the leading coefficient of $f(x)$?

2. Compute $\lim\limits_{x \to 0} \dfrac{e^{x \cos x} - 1 - x}{\sin(x^2)}$.

3. At time 0, an ant is at $(1, 0)$ and a spider is at $(-1, 0)$. The ant starts walking counterclockwise along the unit circle, and the spider starts creeping to the right along the x-axis. It so happens that the ant's horizontal speed is always half the spider's. What will the shortest distance ever between the ant and the spider be?

4. Compute $\displaystyle\sum_{k=1}^{\infty} \frac{k^4}{k!}$.

5. Compute $\displaystyle\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$.

6. A triangle with vertices at $(1003, 0)$, $(1004, 3)$, and $(1005, 1)$ in the xy-plane is revolved all the way around the y-axis. Find the volume of the solid thus obtained.

7. Find all positive real numbers c such that the graph of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - cx$ has the property that the circle of curvature at any local extremum is centered at a point on the x-axis.

8. Compute $\displaystyle\int_0^{\pi/3} x \tan^2(x) dx$.

9. Compute the sum of all real numbers x such that

$$2x^6 - 3x^5 + 3x^4 + x^3 - 3x^2 + 3x - 1 = 0.$$

10. Suppose f and g are differentiable functions such that

$$xg(f(x))f'(g(x))g'(x) = f(g(x))g'(f(x))f'(x)$$

for all real x. Moreover, f is nonnegative and g is positive. Furthermore,

$$\int_0^a f(g(x))dx = 1 - \frac{e^{-2a}}{2}$$

for all reals a. Given that $g(f(0)) = 1$, compute the value of $g(f(4))$.