IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Individual Round: Combinatorics Test

1. Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

2. Compute
$$\sum_{n_{60}=0}^{2} \sum_{n_{59}=0}^{n_{60}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} 1.$$

3. A moth starts at vertex A of a certain cube and is trying to get to vertex B, which is opposite A, in five or fewer "steps," where a step consists in traveling along an edge from one vertex to another. The moth will stop as soon as it reaches B. How many ways can the moth achieve its objective?

4. A dot is marked at each vertex of a triangle ABC. Then, 2, 3, and 7 more dots are marked on the sides AB, BC, and CA, respectively. How many triangles have their vertices at these dots?

5. Fifteen freshmen are sitting in a circle around a table, but the course assistant (who remains standing) has made only six copies of today's handout. No freshman should get more than one handout, and any freshman who does not get one should be able to read a neighbor's. If the freshmen are distinguishable but the handouts are not, how many ways are there to distribute the six handouts subject to the above conditions?

6. For how many ordered triplets (a, b, c) of positive integers less than 10 is the product $a \times b \times c$ divisible by 20?

7. Let n be a positive integer, and let Pushover be a game played by two players, standing squarely facing each other, pushing each other, where the first person to lose balance loses. At the HMPT, 2^{n+1} competitors, numbered 1 through 2^{n+1} clockwise, stand in a circle. They are equals in Pushover: whenever two of them face off, each has a 50% probability of victory. The tournament unfolds in $n+1$ rounds. In each round, the referee randomly chooses one of the surviving players, and the players pair off going clockwise, starting from the chosen one. Each pair faces off in Pushover, and the losers leave the circle. What is the probability that players 1 and 2^n face each other in the last round? Express your answer in terms of n.

8. In how many ways can we enter numbers from the set $\{1, 2, 3, 4\}$ into a 4×4 array so that all of the following conditions hold?

   (a) Each row contains all four numbers.

   (b) Each column contains all four numbers.

   (c) Each "quadrant" contains all four numbers. (The quadrants are the four corner 2×2 squares.)

9. Eight celebrities meet at a party. It so happens that each celebrity shakes hands with exactly two others. A fan makes a list of all unordered pairs of celebrities who shook hands with each other. If order does not matter, how many different lists are possible?

10. Somewhere in the universe, n students are taking a 10-question math competition. Their collective performance is called *laughable* if, for some pair of questions, there exist 57 students such that either all of them answered both questions correctly or none of them answered both questions correctly. Compute the smallest n such that the performance is necessarily laughable.