IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Individual Round: General Test, Part 2

1. Larry can swim from Harvard to MIT (with the current of the Charles River) in 40 minutes, or back (against the current) in 45 minutes. How long does it take him to *row* from Harvard to MIT, if he rows the return trip in 15 minutes? (Assume that the speed of the current and Larry's swimming and rowing speeds relative to the current are all constant.) Express your answer in the format mm:ss.

2. Find
$$\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{4^2}{4^2-1}\cdot\dots\cdot\frac{2006^2}{2006^2-1}.$$

3. Let C be the unit circle. Four distinct, smaller congruent circles C_1, C_2, C_3, C_4 are internally tangent to C such that C_i is externally tangent to C_{i-1} and C_{i+1} for $i = 1, \dots, 4$ where C_5 denotes C_1 and C_0 represents C_4. Compute the radius of C_1.

4. Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

5. If a, b are nonzero real numbers such that $a^2 + b^2 = 8ab$, find the value of $\left|\frac{a+b}{a-b}\right|$.

6. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

7. What is the smallest positive integer n such that n^2 and $(n+1)^2$ both contain the digit 7 but $(n+2)^2$ does not?

8. Six people, all of different weights, are trying to build a human pyramid: that is, they get into the formation

   A

   B C

   D E F

   We say that someone not in the bottom row is "supported by" each of the two closest people beneath her or him. How many different pyramids are possible, if nobody can be supported by anybody of lower weight?

9. Tim has a working analog 12-hour clock with two hands that run continuously (instead of, say, jumping on the minute). He also has a clock that runs really slow—at half the correct rate, to be exact. At noon one day, both clocks happen to show the exact time. At any given instant, the hands on each clock form an angle between 0° and 180° inclusive. At how many times during that day are the angles on the two clocks equal?

10. Fifteen freshmen are sitting in a circle around a table, but the course assistant (who remains standing) has made only six copies of today's handout. No freshman should get more than one handout, and any freshman who does not get one should be able to read a neighbor's. If the freshmen are distinguishable but the handouts are not, how many ways are there to distribute the six handouts subject to the above conditions?