IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Guts Round

IX$^{\text{th}}$ HARVARD-MIT MATHEMATICS TOURNAMENT, 25 FEBRUARY 2006 — GUTS ROUND

1. **[5]** A bear walks one mile south, one mile east, and one mile north, only to find itself where it started. Another bear, more energetic than the first, walks two miles south, two miles east, and two miles north, only to find itself where it started. However, the bears are *not* white and did *not* start at the north pole. At most how many miles apart, to the nearest .001 mile, are the two bears' starting points?

2. **[5]** Compute the positive integer less than 1000 which has exactly 29 positive proper divisors. (Here we refer to positive integer divisors other than the number itself.)

3. **[5]** At a nursey, 2006 babies sit in a circle. Suddenly each baby pokes the baby immediately to either its left or its right, with equal probability. What is the expected number of unpoked babies?

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4. **[6]** Ann and Anne are in bumper cars starting 50 meters apart. Each one approaches the other at a constant ground speed of 10 km/hr. A fly starts at Ann, flies to Anne, then back to Ann, and so on, back and forth until it gets crushed when the two bumper cars collide. When going from Ann to Anne, the fly flies at 20 km/hr; when going in the opposite direction the fly flies at 30 km/hr (thanks to a breeze). How many meters does the fly fly?

5. **[6]** Find the number of solutions in positive integers $(k; a_1, a_2, \ldots, a_k; b_1, b_2, \ldots, b_k)$ to the equation $a_1(b_1) + a_2(b_1 + b_2) + \cdots + a_k(b_1 + b_2 + \cdots + b_k) = 7$.

6. **[6]** Suppose ABC is a triangle such that $AB = 13, BC = 15$, and $CA = 14$. Say D is the midpoint of $\overline{BC}$, E is the midpoint of $\overline{AD}$, F is the midpoint of $\overline{BE}$, and G is the midpoint of $\overline{DF}$. Compute the area of triangle EFG.

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7. **[6]** Find all real numbers x such that $x^2 + \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor = 10$.

8. **[6]** How many ways are there to label the faces of a regular octahedron with the integers 1–8, using each exactly once, so that any two faces that share an edge have numbers that are relatively prime? Physically realizable rotations are considered indistinguishable, but physically unrealizable reflections are considered different.

9. **[6]** Four unit circles are centered at the vertices of a unit square, one circle at each vertex. What is the area of the region common to all four circles?

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10. **[7]** Let $f(x) = x^2 - 2x$. How many distinct real numbers c satisfy $f(f(f(f(c)))) = 3$?

11. **[7]** Find all positive integers $n > 1$ for which $\frac{n^2+7n+136}{n-1}$ is the square of a positive integer.

12. **[7]** For each positive integer n let S_n denote the set $\{1, 2, 3, \ldots, n\}$. Compute the number of triples of subsets A, B, C of S_{2006} (not necessarily nonempty or proper) such that A is a subset of B and $S_{2006} - A$ is a subset of C.

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The problems in this batch all depend on each other. If you solve them correctly, you will produce a triple of mutually consistent answers. There is only one such triple. Your score will be determined by how many of your answers match that triple.

13. **[7]** Let Z be as in problem 15. Let X be the greatest integer such that $|XZ| \leq 5$. Find X.

14. **[7]** Let X be as in problem 13. Let Y be the number of ways to order X crimson flowers, X scarlet flowers, and X vermillion flowers in a row so that no two flowers of the same hue are adjacent. (Flowers of the same hue are mutually indistinguishable.) Find Y.

15. **[7]** Let Y be as in problem 14. Find the maximum Z such that three circles of radius $\sqrt{Z}$ can simultaneously fit inside an equilateral triangle of area Y without overlapping each other.

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16. **[8]** A sequence $a_1, a_2, a_3, \ldots$ of positive reals satisfies $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$. Determine all a_1 such that $a_i = \frac{\sqrt{6}+\sqrt{2}}{4}$ for some positive integer i.

17. **[8]** Begining at a vertex, an ant is crawls between the vertices of a regular octahedron. After reaching a vertex, it randomly picks a neighboring vertex (sharing an edge) and walks to that vertex along the adjoining edge (with all possibilities equally likely.) What is the probability that after walking along 2006 edges, the ant returns to the vertex where it began?

18. **[8]** Cyclic quadrilateral $ABCD$ has side lengths $AB = 1, BC = 2, CD = 3$ and $DA = 4$. Points P and Q are the midpoints of $\overline{BC}$ and $\overline{DA}$. Compute PQ^2.

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19. **[8]** Let ABC be a triangle with $AB = 2, CA = 3, BC = 4$. Let D be the point diametrically opposite A on the circumcircle of ABC, and let E lie on line AD such that D is the midpoint of $\overline{AE}$. Line l passes through E perpendicular to $\overline{AE}$, and F and G are the intersections of the extensions of $\overline{AB}$ and $\overline{AC}$ with l. Compute FG.

20. **[8]** Compute the number of real solutions (x, y, z, w) to the system of equations:

$$x = z + w + zwx \qquad z = x + y + xyz$$
$$y = w + x + wxy \qquad w = y + z + yzw$$

21. **[8]** Find the smallest positive integer k such that $z^{10} + z^9 + z^6 + z^5 + z^4 + z + 1$ divides $z^k - 1$.

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22. **[9]** Let $f(x)$ be a degree 2006 polynomial with complex roots $c_1, c_2, \ldots, c_{2006}$, such that the set $\{|c_1|, |c_2|, \ldots, |c_{2006}|\}$ consists of exactly 1006 distinct values. What is the minimum number of real roots of $f(x)$?

23. **[9]** Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers defined by $a_0 = 21, a_1 = 35$, and $a_{n+2} = 4a_{n+1} - 4a_n + n^2$ for $n \geq 2$. Compute the remainder obtained when a_{2006} is divided by 100.

24. **[9]** Two 18-24-30 triangles in the plane share the same circumcircle as well as the same incircle. What's the area of the region common to both the triangles?

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25. **[9]** Points A, C, and B lie on a line in that order such that $AC = 4$ and $BC = 2$. Circles ω_1, ω_2, and ω_3 have $\overline{BC}, \overline{AC}$, and $\overline{AB}$ as diameters. Circle Γ is externally tangent to ω_1 and ω_2 at D and E respectively, and is internally tangent to ω_3. Compute the circumradius of triangle CDE.

26. **[9]** Let $a \geq b \geq c$ be real numbers such that

$$\begin{aligned} a^2bc + ab^2c + abc^2 + 8 &= a + b + c \\ a^2b + a^2c + b^2c + b^2a + c^2a + c^2b + 3abc &= -4 \\ a^2b^2c + ab^2c^2 + a^2bc^2 &= 2 + ab + bc + ca \end{aligned}$$

If $a + b + c > 0$, then compute the integer nearest to a^5.

27. **[9]** Let N denote the number of subsets of $\{1, 2, 3, \ldots, 100\}$ that contain more prime numbers than multiples of 4. Compute the largest integer k such that 2^k divides N.

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28. **[10]** A pebble is shaped as the intersection of a cube of side length 1 with the solid sphere tangent to all of the cube's edges. What is the surface area of this pebble?

29. **[10]** Find the area in the first quadrant bounded by the hyperbola $x^2 - y^2 = 1$, the x-axis, and the line $3x = 4y$.

30. **[10]** ABC is an acute triangle with incircle ω. ω is tangent to sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively. P is a point on the altitude from A such that Γ, the circle with diameter $\overline{AP}$, is tangent to ω. Γ intersects $\overline{AC}$ and $\overline{AB}$ at X and Y respectively. Given $XY = 8$, $AE = 15$, and that the radius of Γ is 5, compute $BD \cdot DC$.

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The problems in this batch all depend on each other. If you solve them correctly, you will produce a triple of mutually consistent answers. There is only one such triple. Your score will be determined by how many of your answers match that triple.

31. **[10]** Let A be as in problem 33. Let W be the sum of all positive integers that divide A. Find W.

32. **[10]** In the alphametic $WE \times EYE = SCENE$, each different letter stands for a different digit, and no word begins with a 0. The W in this problem has the same value as the W in problem 31. Find S.

33. **[10]** Let W, S be as in problem 32. Let A be the least positive integer such that an acute triangle with side lengths S, A, and W exists. Find A.

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34. **[12]** In bridge, a standard 52-card deck is dealt in the usual way to 4 players. By convention, each hand is assigned a number of "points" based on the formula

$$4 \times (\#\text{ A's}) + 3 \times (\#\text{ K's}) + 2 \times (\#\text{ Q's}) + 1 \times (\#\text{ J's}).$$

Given that a particular hand has exactly 4 cards that are A, K, Q, or J, find the probability that its point value is 13 or higher.

35. **[12]** A sequence is defined by $A_0 = 0, A_1 = 1, A_2 = 2$, and, for integers $n \geq 3$,

$$A_n = \frac{A_{n-1} + A_{n-2} + A_{n-3}}{3} + \frac{1}{n^4 - n^2}$$

Compute $\lim_{N\to\infty} A_N$.

36. **[12]** Four points are independently chosen uniformly at random from the interior of a regular dodecahedron. What is the probability that they form a tetrahedron whose interior contains the dodecahedron's center?

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37. **[15]** Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{2^n \cdot (n^3+7n^2+14n+8)}$

38. **[15]** Suppose ABC is a triangle with incircle ω, and ω is tangent to $\overline{BC}$ and $\overline{CA}$ at D and E respectively. The bisectors of $\angle A$ and $\angle B$ intersect line DE at F and G respectively, such that $BF = 1$ and $FG = GA = 6$. Compute the radius of ω.

39. **[15]** A *fat coin* is one which, when tossed, has a 2/5 probability of being heads, 2/5 of being tails, and 1/5 of landing on its edge. Mr. Fat starts at 0 on the real line. Every minute, he tosses a fat coin. If it's heads, he moves left, decreasing his coordinate by 1; if it's tails, he moves right, increasing his coordinate by 1. If the coin lands on its edge, he moves back to 0. If Mr. Fat does this *ad infinitum*, what fraction of his time will he spend at 0?

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40. **[18]** Compute $\displaystyle\sum_{k=1}^{\infty} \frac{3k+1}{2k^3+k^2} \cdot (-1)^{k+1}$.

41. **[18]** Let Γ denote the circumcircle of triangle ABC. Point D is on $\overline{AB}$ such that $\overline{CD}$ bisects $\angle ACB$. Points P and Q are on Γ such that $\overline{PQ}$ passes through D and is perpendicular to $\overline{CD}$. Compute PQ, given that $BC = 20, CA = 80, AB = 65$.

42. **[18]** Suppose hypothetically that a certain, very corrupt political entity in a universe holds an election with two candidates, say A and B. A total of 5,825,043 votes are cast, but, in a sudden rainstorm, all the ballots get soaked. Undaunted, the election officials decide to guess what the ballots say. Each ballot has a 51% chance of being deemed a vote for A, and a 49% chance of being deemed a vote for B. The probability that B will win is 10^{-X}. What is X *rounded to the nearest* 10?

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43. Write down at least one, and up to ten, different 3-digit prime numbers. If you somehow fail to do this, we will ignore your submission for this problem. Otherwise, you're entered into a game with other teams. In this game, you start with 10 points, and each number you write down is like a bet: if no one else writes that number, you gain 1 point, but if anyone else writes that number, you lose 1 point. Thus, your score on this problem can be anything from 0 to 20.

44. On the Euclidean plane are given 14 points:

$A = (0, 428)$	$B = (9, 85)$	$C = (42, 865)$	$D = (192, 875)$
$E = (193, 219)$	$F = (204, 108)$	$G = (292, 219)$	$H = (316, 378)$
$I = (375, 688)$	$J = (597, 498)$	$K = (679, 766)$	$L = (739, 641)$
	$M = (772, 307)$	$N = (793, 0)$	

A fly starts at A, visits all the other points, and comes back to A in such a way as to minimize the total distance covered. What path did the fly take? Give the names of the points it visits in order. Your score will be

$$20 + \lfloor \text{the optimal distance} \rfloor - \lfloor \text{your distance} \rfloor$$

or 0, whichever is greater.

45. On your answer sheet, *clearly* mark at least seven points, as long as

(i) No three are collinear.

(ii) No seven form a convex heptagon.

Please do not cross out any points; erase if you can do so neatly. If the graders deem that your paper is too messy, or if they determine that you violated one of those conditions, your submission for this problem will be disqualified. Otherwise, your score will be the number of points you marked minus 6, even if you actually violated one of the conditions but were able to fool the graders.