IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament
Saturday 25 February 2006

Team Round A

Mobotics [120]

Spring is finally here in Cambridge, and it's time to mow our lawn. For the purpose of these problems, our lawn consists of little *clumps* of grass arranged at the points of a certain grid (to be specified later). Our machinery consists of a fleet of identical *mowbots* (or "mobots" for short). A mobot is a lawn-mowing machine. To mow our lawn, we begin by choosing a *formation*: we place as many mobots as we want at various clumps of grass and orient each mobot's head in a certain direction. At the blow of a whistle, each mobot starts moving in the direction we've chosen, mowing every clump of grass in its path (including the clump it starts on) until it goes off the lawn.

Because the spring is so young, our lawn is rather delicate. Consequently, we want to make sure that every clump of grass is mowed once and only once. We will not consider formations that do not meet this criterion.

One more thing: two formations are considered "different" if there exists a clump of grass for which either (1) for exactly one of the formations does a mobot starts on that clump, or (2) there are mobots starting on this clump for both the formations, but they're oriented in different directions.

1. **[15]** For this problem, our lawn consists of a row of n clumps of grass. This row runs in an east-west direction. In our formation, each mobot may be oriented toward the north, south, east, or west. One example of an allowable formation if $n = 6$ is symbolized below:

$$\cdot \quad \cdot \leftarrow \quad \uparrow \quad \uparrow \quad \downarrow$$

(The mobot on the third clump will move westward, mowing the first three clumps. Each of the last three clumps is mowed by a different mobot.) Here's another allowable formation for $n = 6$, considered different from the first:

$$\cdot \quad \cdot \leftarrow \quad \uparrow \quad \uparrow \quad \rightarrow$$

Compute the number of different allowable formations for any given n.

2. **[25]** For this problem, our lawn is an $m \times n$ rectangular grid of clumps, that is, with m rows running east-west and n columns running north-south. To be even more explicit, we might say our clumps are at the lattice points

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \le x < n \text{ and } 0 \le y < m\}.$$

However, mobots are now allowed to be oriented to go either north or east only. So one allowable formation for $m = 2$, $n = 3$ might be as follows:

$$\begin{array}{ccc} \cdot & \rightarrow & \cdot \\ \uparrow & \rightarrow & \cdot \end{array}$$

Prove that the number of allowable formations for given m and n is $\dfrac{(m+n)!}{m!\,n!}$.

3. **[40]** In this problem, we stipulate that $m \ge n$, and the lawn is shaped differently. The clumps are now at the lattice points in a trapezoid:

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \le x < n \text{ and } 0 \le y < m + 1 - n + x\},$$

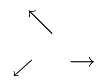
As in problem 2, mobots can be set to move either north or east. For given m and n, determine with proof the number of allowable formations.

4. **[15]** In this problem and the next, the lawn consists of points in a triangular grid of size n, so that for $n = 3$ the lawn looks like

$$\begin{array}{ccccc} & & \cdot & & \\ & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot \end{array}$$

Mobots are allowed to be oriented to the east, 30° west of north, or 30° west of south. Under these conditions, for any given n, what is the minimum number of mobots needed to now the lawn?

5. **[25]** With the same lawn and the same allowable mobot orientations as in the previous problem, let us call a formation "happy" if it is invariant under 120° rotations. (A rotation applies both to the positions of the mobots and to their orientations.) An example of a happy formation for $n = 2$ might be



Find the number of happy formations for a given n.

Polygons [110]

6. **[15]** Let n be an integer at least 5. At most how many diagonals of a regular n-gon can be simultaneously drawn so that no two are parallel? Prove your answer.

7. **[25]** Given a convex n-gon, $n \geq 4$, at most how many diagonals can be drawn such that each drawn diagonal intersects every other drawn diagonal either in the interior of the n-gon or at a vertex? Prove your answer.

8. **[15]** Given a regular n-gon with sides of length 1, what is the smallest radius r such that there is a non-empty intersection of n circles of radius r centered at the vertices of the n-gon? Give r as a formula in terms of n. Be sure to prove your answer.

9. **[40]** Let $n \geq 3$ be a positive integer. Prove that given any n angles $0 < \theta_1, \theta_2, \ldots, \theta_n < 180^\circ$, such that their sum is $180(n-2)$ degrees, there exists a convex n-gon having exactly those angles, in that order.

10. **[15]** Suppose we have an n-gon such that each interior angle, measured in degrees, is a positive integer. Suppose further that all angles are less than 180°, and that all angles are different sizes. What is the maximum possible value of n? Prove your answer.

What do the following problems have in common? [170]

11. **[15]** The lottery cards of a certain lottery contain all nine-digit numbers that can be formed with the digits 1, 2 and 3. There is exactly one number on each lottery card. There are only red, yellow and blue lottery cards. Two lottery numbers that differ from each other in all nine digits always appear on cards of different color. Someone draws a red card and a yellow card. The red card has the number 122 222 222 and the yellow card has the number 222 222 222. The first prize goes to the lottery card with the number 123 123 123. What color(s) can it possibly have? Prove your answer.

12. **[25]** A $3 \times 3 \times 3$ cube is built from 27 unit cubes. Suddenly five of those cubes mysteriously teleport away. What is the minimum possible surface area of the remaining solid? Prove your answer.

13. **[40]** Having lost a game of checkers and my temper, I dash all the pieces to the ground but one. This last checker, which is perfectly circular in shape, remains completely on the board, and happens to cover equal areas of red and black squares. Prove that the center of this piece must lie on a boundary between two squares (or at a junction of four).

14. **[40]** A number n is called *bummed out* if there is exactly one ordered pair of positive integers (x, y) such that
$$\lfloor x^2/y \rfloor + \lfloor y^2/x \rfloor = n.$$
Find all bummed out numbers.

15. **[50]** Find, with proof, all positive integer palindromes whose square is also a palindrome.