Calculus Solutions

Harvard-MIT Math Tournament
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Problem C1 [3 points]

Find all twice differentiable functions $f(x)$ such that $f''(x) = 0$, $f(0) = 19$, and $f(1) = 99$.

Solution: Since $f''(x) = 0$ we must have $f(x) = ax + b$ for some real numbers a, b. Thus $f(0) = b = 19$ and $f(1) = a + 19 = 99$, so $a = 80$. Therefore $f(x) = \mathbf{80x + 19}$.

Problem C2 [3 points]

A rectangle has sides of length $\sin x$ and $\cos x$ for some x. What is the largest possible area of such a rectangle?

Solution: We wish to maximize $\sin x \cdot \cos x = \frac{1}{2}\sin 2x$. But $\sin 2x \leq 1$, with equality holding for $x = \pi/4$, so the maximum is $\mathbf{\frac{1}{2}}$.

Problem C3 [4 points]

Find

$$\int_{-4\pi\sqrt{2}}^{4\pi\sqrt{2}} \left(\frac{\sin x}{1+x^4} + 1\right) dx.$$

Solution: The function $\frac{\sin x}{1+x^4}$ is odd, so its integral over this interval is 0. Thus we get the same answer if we just integrate dx, namely, $\mathbf{8}\pi\sqrt{\mathbf{2}}$.

Problem C4 [4 points]

f is a continuous real-valued function such that $f(x+y) = f(x)f(y)$ for all real x, y. If $f(2) = 5$, find $f(5)$.

Solution 1: Since $f(nx) = f(x)^n$ for all integers n, $f(5) = f(1)^5$ and $f(2) = f(1)^2$, so $f(5) = f(2)^{5/2} = \mathbf{25}\sqrt{\mathbf{5}}$.

Solution 2: More generally, since $f(nx) = f(x)^n$ for all integers n, $f(1) = c = f(1/n)^n$ for some constant c and all integers n. Thus $f(k/n) = f(1/n)^k = f(1)^{k/n} = c^{k/n}$ for all rational numbers k/n. By continuity, it follows that $f(x) = c^x$ for all real numbers x. Since $f(2) = 5$, $c = \sqrt{5}$, so $f(5) = \mathbf{25}\sqrt{\mathbf{5}}$.

Problem C5 [5 points]

Let $f(x) = x + \cfrac{1}{2x+\cfrac{1}{2x+\cfrac{1}{2x+\ddots}}}$ for $x > 0$. Find $f(99)f'(99)$.

Solution: Assume that the continued fraction converges (it does) so that $f(x)$ is well defined. Notice that $f(x) - x = \frac{1}{x+f(x)}$, so $f(x)^2 - x^2 = 1$, or $f(x) = \sqrt{1+x^2}$ (we need the positive square root since $x > 0$). Thus $f'(x) = \frac{x}{\sqrt{1+x^2}}$, so $f(x)f'(x) = x$. In particular, $f(99)f'(99) = \mathbf{99}$.

Problem C6 [5 points]

Evaluate $\frac{d}{dx}(\sin x - \frac{4}{3}\sin^3 x)$ when $x = 15$.

Solution: Of course this problem can be done by brute force, differentiating and then using the half angle formula to find sin and cos of 15, but there is a quicker way. $e^{ix} = \cos x + i \sin x$, so $sin(3x)$ is the imaginary part of $(\cos x + i\sin x)^3$, which is $3\cos^2 x \sin x - \sin^3 x = 3\sin x - 4\sin^3 x$, so the expression we are differentiating is just $\frac{1}{3}\sin(3x)$. Hence the derivative is $\cos(3x)$, and $\cos 45 = \frac{\sqrt{\mathbf{2}}}{\mathbf{2}}$.

Problem C7 [5 points]

If a right triangle is drawn in a semicircle of radius 1/2 with one leg (not the hypotenuse) along the diameter, what is the triangle's maximum possible area?

Solution: It is easy to see that we will want one vertex of the triangle to be where the diameter meets the semicircle, so the diameter is divided into segments of length x and $1-x$, where x is the length of the leg on the diameter. The other leg of the triangle will be the geometric mean of these two numbers, $\sqrt{x(1-x)}$. Therefore the area of the triangle is $\frac{x\sqrt{x(1-x)}}{2}$, so it will be maximized when $\frac{d}{dx}(x^3 - x^4) = 3x^2 - 4x^3 = 0$, or $x = 3/4$. Therefore the maximum area is $\frac{\mathbf{3}\sqrt{\mathbf{3}}}{\mathbf{32}}$.

Problem C8 [6 points]

A circle is randomly chosen in a circle of radius 1 in the sense that a point is randomly chosen for its center, then a radius is chosen at random so that the new circle is contained in the original circle. What is the probability that the new circle contains the center of the original circle?

Solution: If the center of the new circle is more than 1/2 away from the center of the original circle then the new circle cannot possibly contain the center of the original one. Let x be the distance between the centers (by symmetry this is all we need to consider), then for $0 \le x \le 1/2$ the probability of the new circle containing the center of the original one is $1 - \frac{x}{1-x}$. Hence we need to compute $\int_0^{1/2} (1 - \frac{x}{1-x})dx = \frac{1}{2} - \int_0^{1/2} \frac{x}{1-x}dx$. To evaluate the integral, we can integrate by parts to get

$$-x\ln(1-x)|_0^{1/2} - \int_0^{1/2} -\ln(1-x)dx = -\frac{1}{2}\ln(\frac{1}{2}) - [(1-x)\ln(1-x) - (1-x)]_0^{1/2} = \ln 2 - \frac{1}{2}.$$

Alternatively, we can use polynomial division to find that $\frac{x}{1-x} = -1 + \frac{1}{1-x}$, so $\int_0^{1/2} \frac{x}{1-x} dx = \int_0^{1/2} (-1 + \frac{1}{1-x}) dx = \ln 2 - \frac{1}{2}$. Therefore the probability is $\frac{1}{2} - (\ln 2 - \frac{1}{2}) = \mathbf{1 - \ln 2}$.

Problem C9 [7 points]

What fraction of the Earth's volume lies above the 45 degrees north parallel? You may assume the Earth is a perfect sphere. The volume in question is the smaller piece that we would get if the sphere were sliced into two pieces by a plane.

Solution 1: Without loss of generality, look at a sphere of radius 1 centered at the origin. If you like cartesian coordinates, then you can slice the sphere into discs with the same z coordinate, which have radius $\sqrt{1-z^2}$, so the region we are considering has volume $\int_{\sqrt{2}/2}^{1} \pi(1-z^2)dz = \pi(\frac{2}{3} - \frac{5\sqrt{2}}{12})$, and dividing by $4\pi/3$ we get $\mathbf{\frac{8-5\sqrt{2}}{16}}$.

Solution 2: For those who prefer spherical coordinates, we can find the volume of the spherical cap plus a cone whose vertex is the center of the sphere. This region is where r ranges from 0 to 1, θ ranges from 0 to 2π, and ϕ ranges from 0 to $\pi/4$. Remembering we need to subtract off the volume of the cone, which has height $\frac{1}{\sqrt{2}}$ and a circular base of radius $\frac{1}{\sqrt{2}}$, then divide by $\frac{4}{3}\pi$ to get the fraction of the volume of the sphere, we find that we need to evaluate $\frac{\int_0^1 \int_0^{2\pi} \int_0^{\pi/4} r^2 \sin\phi d\phi d\theta dr - \frac{1}{3}\pi(\frac{1}{\sqrt{2}})^2 \frac{1}{\sqrt{2}}}{4\pi/3}$. The integral is just $2\pi\frac{1}{3}(-\cos\pi/4 + \cos 0) = \frac{4\pi - 2\pi\sqrt{2}}{6}$. Putting this back in the answer and simplifying yields $\mathbf{\frac{8-5\sqrt{2}}{16}}$.

Solution 3: Cavalieri's Principle states that if two solids have the same cross-sectional areas at every height, then they have the same volume. This principle is very familiar in the plane, where we know that the area of a triangle depends only on the base and height, not the precise position of the apex. To apply it to a sphere, consider a cylinder with radius 1 and height 1. Now cut out a cone whose base is the upper circle of the cylinder and whose apex is the center of the lower circle. Then at a height z the area is $\pi(1^2 - z^2)$, exactly the same as for the upper hemisphere! The portion lying above the the 45 degrees north parallel is that which ranges from height $\frac{1}{\sqrt{2}}$ to 1. The volume of the cylinder in this range is $\pi \cdot 1^2(1 - \frac{1}{\sqrt{2}})$. The volume of the cone in this range is the volume of the entire cone minus the portion from height 0 to $\frac{1}{\sqrt{2}}$, i.e., $\frac{1}{3}\pi(1^2 \cdot 1 - (\frac{1}{\sqrt{2}})^2 \frac{1}{\sqrt{2}})$. Therefore the fraction of the Earth's volume that lies above the 45 degrees north parallel is $\frac{\pi(1-\frac{1}{\sqrt{2}}) - \frac{1}{3}\pi(1-\frac{1}{2\sqrt{2}})}{4\pi/3} = \mathbf{\frac{8-5\sqrt{2}}{16}}$.

Solution 4: Another way to approach this problem is to integrate the function $\sqrt{1-x^2-y^2}$ over the region $x^2 + y^2 \leq \frac{1}{\sqrt{2}}$, subtract off a cylinder of radius and height $\frac{1}{\sqrt{2}}$, then divide by the volume of the sphere. One could also use the nontrivial fact that the surface area of a portion of a sphere of radius r between two parallel planes separated by a distance z is $2\pi r^2 z$, so in particular the surface area of this cap is $2\pi(1 - \frac{1}{\sqrt{2}})$. Now, the ratio of the surface area of the cap to the surface area of the sphere is the same as the ratio of the volume of the cap plus the cone considered in Solution 2 to the volume of the whole sphere, so this allows us to avoid integration entirely.

Problem C10 [8 points]

Let A_n be the area outside a regular n-gon of side length 1 but inside its circumscribed circle, let B_n be the area inside the n-gon but outside its inscribed circle. Find the limit as n tends to infinity of $\frac{A_n}{B_n}$.

Solution: The radius of the inscribed circle is $\frac{1}{2}\cot\frac{\pi}{n}$, the radius of the circumscribed circle is $\frac{1}{2}\csc\frac{\pi}{n}$, and the area of the n-gon is $\frac{n}{4}\cot\frac{\pi}{n}$. The diagram below should help you verify that these are correct.

Then $A_n = \pi(\frac{1}{2}\csc\frac{\pi}{n})^2 - \frac{n}{4}\cot\frac{\pi}{n}$, and $B_n = \frac{n}{4}\cot\frac{\pi}{n} - \pi(\frac{1}{2}\cot\frac{\pi}{n})^2$, so $\frac{A_n}{B_n} = \frac{\pi(\csc\frac{\pi}{n})^2 - n\cot\frac{\pi}{n}}{n\cot\frac{\pi}{n} - \pi(\cot\frac{\pi}{n})^2}$. Let s denote $\sin\frac{\pi}{n}$ and c denote $\cos\frac{\pi}{n}$. Multiply numerator and denominator by s^2 to get $\frac{A_n}{B_n} = \frac{\pi - ncs}{ncs - \pi c^2}$. Now use Taylor series to replace s by $\frac{\pi}{n} - \frac{(\frac{\pi}{n})^3}{6} + \ldots$ and c by $1 - \frac{(\frac{\pi}{n})^2}{2} + \ldots$. By l'Hôpital's rule it will suffice to take just enough terms so that the highest power of n in the numerator and denominator is determined, and that turns out to be n^{-2} in each case. In particular, we get the limit $\frac{A_n}{B_n} = \frac{\pi - n\frac{\pi}{n} + n\frac{2}{3}(\frac{\pi}{n})^3 + \ldots}{n\frac{\pi}{n} - n\frac{2}{3}(\frac{\pi}{n})^3 - \pi + \pi(\frac{\pi}{n})^2 + \ldots} = \frac{\frac{2}{3}\frac{\pi^3}{n^2} + \ldots}{\frac{1}{3}\frac{\pi^3}{n^2} + \ldots} \to \mathbf{2}$.