Algebra Test Solutions

Harvard-MIT Math Tournament

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1. Find $x - y$, given that $x^4 = y^4 + 24$, $x^2 + y^2 = 6$, and $x + y = 3$.
Solution: $\frac{24}{6\cdot 3} = \frac{x^4 - y^4}{(x^2+y^2)(x+y)} = \frac{(x^2+y^2)(x+y)(x-y)}{(x^2+y^2)(x+y)} = x - y = \boxed{\frac{4}{3}}$.

2. Find $(x+1)(x^2+1)(x^4+1)(x^8+1)\cdots$, where $|x| < 1$.
Solution: Let $S = (x+1)(x^2+1)(x^4+1)(x^8+1)\cdots = 1 + x + x^2 + x^3 + \cdots$. Since $xS = x + x^2 + x^3 + x^4 + \cdots$, we have $(1-x)S = 1$, so $S = \boxed{\frac{1}{1-x}}$.

3. How many times does 24 divide into 100! (factorial)?
Solution: We first determine the number of times 2 and 3 divide into $100! = 1\cdot 2\cdot 3\cdots 100$. Let $\langle N\rangle_n$ be the number of times n divides into N (i.e. we want to find $\langle 100!\rangle_{24}$). Since 2 only divides into even integers, $\langle 100!\rangle_2 = \langle 2\cdot 4\cdot 6\cdots 100\rangle$. Factoring out 2 once from each of these multiples, we get that $\langle 100!\rangle_2 = \langle 2^{50}\cdot 1\cdot 2\cdot 3\cdots 50\rangle_2$. Repeating this process, we find that $\langle 100!\rangle_2 = \langle 20^{50+25+12+6+3+1}\cdot 1\rangle_2 = 97$. Similarly, $\langle 100!\rangle_3 = \langle 3^{33+11+3+1}\rangle_3 = 48$. Now $24 = 2^3\cdot 3$, so for each factor of 24 in 100! there needs to be three multiples of 2 and one multiple of 3 in 100!. Thus $\langle 100!\rangle_{24} = ([\langle 100!\rangle_2/3] + \langle 100!\rangle_3) = \boxed{32}$, where $[N]$ is the greatest integer less than or equal to N.

4. Given that $7,999,999,999$ has at most two prime factors, find its largest prime factor.
Solution: $7,999,999,999 = 8\cdot 10^9 - 1 = 2000^3 - 1 = (2000-1)(2000^2 + 2000 + 1)$, so $(2000^2 + 2000 + 1) = \boxed{4{,}002{,}001}$ is its largest prime factor.

5. Find the 6-digit number beginning and ending in the digit 2 that is the product of three consecutive even integers.
Solution: Because the last digit of the product is 2, none of the three consecutive even integers end in 0. Thus they must end in 2, 4, 6 or 4, 6, 8, so they must end in 4, 6, 8 since $2\cdot 4\cdot 6$ does not end in 2. Call the middle integer n. Then the product is $(n-2)n(n+2) = n^3 - 4n$, so $n > \sqrt[3]{200000} = \sqrt[3]{200\cdot 10^3} \approx 60$, but clearly $n < \sqrt[3]{300000} = \sqrt[3]{300\cdot 10^3} < 70$. Thus $n = 66$, and the product is $66^3 - 4\cdot 66 = \boxed{287232}$.

6. What is the last digit of $1^1 + 2^2 + 3^3 + \cdots + 100^{100}$?
Solution: Let $L(d,n)$ be the last digit of a number ending in d to the nth power. For $n \geq 1$, we know that $L(0,n) = 0$, $L(1,n) = 1$, $L(5,n) = 5$, $L(6,n) = 6$. All numbers ending in odd digits in this series are raised to odd powers; for odd n, $L(3,n) = 3$ or 7, $L(7,n) = 3$ or 7, $L(9,n) = 9$. All numbers ending in even digits are raised to even powers; for even n, $L(2,n) = 4$ or 6, $L(4,n) = L(6,n) = 6$, $L(8,n) = 6$ or 4. Further, for each

last digit that has two possible values, the possible values will be present equally as often. Now define $S(d)$ such that $S(0)$=0 and for $1 \leq d \leq 9$, $S(d) = L(d,d) + L(d,d+10) + L(d,d+20) + L(d,d+30) + \cdots + L(d,d+90)$, so that the sum we want to calculate becomes $S(0) + S(1) + S(2) + \cdots + S(9)$. But by the above calculations all $S(d)$ are divisible by 10, so their sum is divisible by 10, which means its last digit is $\boxed{0}$.

7. A polynomial P has four roots, $\frac{1}{4}, \frac{1}{2}, 2, 4$. The product of the roots is 1, and $P(1) = 1$. Find $P(0)$.

Solution: A polynomial Q with n roots, $x_1, \ldots, x_n$, and $Q(x_0) = 1$ is given by $Q(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_4)}$, so $P(0) = \frac{1}{\frac{3}{4}\cdot\frac{1}{2}\cdot(-1)\cdot(-3)} = \boxed{\frac{8}{9}}$.

8. How many integers between 1 and 2000 inclusive share no common factors with 2001?

Solution: Two integers are said to be *relatively prime* if they share no common factors, that is if there is no integer greater than 1 that divides evenly into both of them. Note that 1 is relatively prime to all integers. Let $\varphi(n)$ be the number of integers less than n that are relatively prime to n. Since $\varphi(mn) = \varphi(m)\varphi(n)$ for m and n relatively prime , we have $\varphi(2001) = \varphi(3 \cdot 23 \cdot 29) = (3-1)(23-1)(29-1) = \boxed{1232}$.

9. Find the number of positive integer solutions to $n^x + n^y = n^z$ with $n^z < 2001$.

Solution: If $n = 1$, the relation cannot hold, so assume otherwise. If $x > y$, the left hand side factors as $n^y(n^{x-y} + 1)$ so $n^{x-y} + 1$ is a power of n. But it leaves a remainder of 1 when divided by n and is greater than 1, a contradiction. We reach a similar contradiction if $y > x$. So $y = x$ and $2n^x = n^z$, so 2 is a power of n and $n = 2$. So all solutions are of the form $2^x + 2^x = 2^{x+1}$, which holds for all x. $2^{x+1} < 2001$ implies $x < 11$, so there are $\boxed{10}$ solutions.

10. Find the real solutions of $(2x+1)(3x+1)(5x+1)(30x+1) = 10$.

Solution: $(2x+1)(3x+1)(5x+1)(30x+1) = [(2x+1)(30x+1)][(3x+1)(5x+1)] = (60x^2+32x+1)(15x^2+8x+1) = (4y+1)(y+1) = 10$, where $y = 15x^2+8x$. The quadratic equation in y yields $y = 1$ and $y = -\frac{9}{4}$. For $y = 1$, we have $15x^2+8x-1 = 0$, so $x = \frac{-4\pm\sqrt{31}}{15}$. For $y = -\frac{9}{4}$, we have $15x^2+8x+\frac{9}{4}$, which yields only complex solutions for x. Thus the real solutions are $\boxed{\frac{-4\pm\sqrt{31}}{15}}$.