Team Round Solutions
Harvard-MIT Math Tournament
March 3, 2001

1. How many digits are in the base two representation of 10! (factorial)?

Solution: We write $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. The number of digits (base 2) of 10! is equal to $[\log_2 10!] = 8 + \log_2(3^4 \cdot 5^2 \cdot 7)$. Since $2^13 < 3^2 \cdot 5^2 \cdot 7 < 2^14$, the number of digits is $8 + 13 = \boxed{21}$.

2. On a certain unidirectional highway, trucks move steadily at 60 miles per hour spaced 1/4 of a mile apart. Cars move steadily at 75 miles per hour spaced 3 seconds apart. A lone sports car weaving through traffic at a steady forward speed passes two cars between each truck it passes. How quickly is it moving in miles per hour?

Solution: The cars are 1/8 of a mile apart. Consider the reference frame in which the trucks move at 0 velocity (and the cars move at 15). Call the speed of the sports car in this reference frame v. The amount of time for the sports car to move from one truck to the next is $\frac{1/4 \text{ miles}}{v}$, and the amount of time for two regular cars to pass the truck is $\frac{1/8 \text{ miles}}{15\text{mph}}$. Equating these, we get $v = 30$, and $v + 60 = \boxed{90}$ mph.

3. What is the 18th digit after the decimal point of $\frac{10000}{9899}$?

Solution: $\frac{10000}{9899}$ satisfies $100(x-1) = 1.01x$, so each pair of adjacent digits is generated by adding the previous two pairs of digits. So the decimal is $1.01020305081321345590\ldots$, and the 18th digit is $\boxed{5}$.

4. P is a polynomial. When P is divided by $x-1$, the remainder is -4. When P is divided by $x-2$, the remainder is -1. When P is divided by $x-3$, the remainder is 4. Determine the remainder when P is divided by $x^3 - 6x^2 + 11x - 6$.

Solution: The remainder polynomial is simply the order two polynomial that goes through the points $(1,-4)$, $(2,-1)$, and $(3,4)$: $\boxed{x^2 - 5}$.

5. Find all x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that $1 - \sin^4 x - \cos^2 x = \frac{1}{16}$.

Solution: $1 - \sin^4 x - \cos^2 x = \frac{1}{16} \Rightarrow (16 - 16\cos^2 x) - \sin^4 x - 1 = 0 \Rightarrow 16\sin^4 x - 16\sin^2 x + 1 = 0$. Use the quadratic formula in $\sin x$ to obtain $\sin^2 x = \frac{1}{2} \pm \frac{\sqrt{3}}{4}$. Since $\cos 2x = 1 - 2\sin^2 x = \pm\frac{\sqrt{3}}{2}$, we get $x = \boxed{\pm\frac{\pi}{12}, \pm\frac{5\pi}{12}}$.

6. What is the radius of the smallest sphere in which 4 spheres of radius 1 will fit?

Solution: The centers of the smaller spheres lie on a tetrahedron. Let the points of the tetrahedron be $(1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$, and $(1,-1,-1)$. These points have distance $\sqrt(3)$ from the center, and $\sqrt(2)$ from each other, so the radius of the smallest sphere in which 4 spheres of radius $\sqrt(2)$ will fit is $\sqrt(2)+\sqrt(3)$. Scale this to the correct answer by dividing by $\sqrt(2)$: $\boxed{\frac{2+\sqrt{6}}{2}}$.

7. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. The Lucas numbers are defined by $L_1 = 1$, $L_2 = 2$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$. Calculate $\frac{\prod_{n=1}^{15} \frac{F_{2n}}{F_n}}{\prod_{n=1}^{13} L_n}$.

Solution: It is easy to show that $L_n = \frac{F_{2n}}{F_n}$, so the product above is $L_1 4 L_1 5 = 843 \cdot 1364 = \boxed{1149852}$.

8. Express $\frac{\sin 10+\sin 20+\sin 30+\sin 40+\sin 50+\sin 60+\sin 70+\sin 80}{\cos 5\cos 10\cos 20}$ without using trigonometric functions.

Solution: We will use the identities $\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$ and $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a+b}{2}$. The numerator is $(\sin 10+\sin 80)+(\sin 20+\sin 70)+(\sin 30+\sin 60)+(\sin 40+\sin 60) = 2\sin 45\,(\cos 35+\cos 25+\cos 15+\cos 35) = 2\sin 45\,((\cos 35+\cos 5)+(\cos 25+\cos 15)) = 4\sin 45\cos 20\,(\cos 15+\cos 5) = 8\sin 45\cos 20\cos 10\cos 5$, so the fraction equals $8\sin 45 = \boxed{4\sqrt{2}}$.

9. Compute $\sum_{i=1}^{\infty}\frac{ai}{a^i}$ for $a > 1$.

Solution: The sum $S = a + ax + ax^2 + ax^3 + \cdots$ for $x < 1$ can be determined by realizing that $xS = ax + ax^2 + ax^3 + \cdots$ and $(1-x)S = a$, so $S = \frac{a}{1-x}$. Using this, we have $\sum_{i=1}^{\infty}\frac{ai}{a^i} = a\sum_{i=1}^{\infty}\frac{i}{a^i} = a\left[\frac{1}{a}+\frac{2}{a^2}+\frac{3}{a^3}+\cdots\right] = a\left[\left(\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\cdots\right)+\left(\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\cdots\right)+\cdots\right] = a\left[\frac{1}{1-a}+\frac{1}{a}\frac{1}{1-a}+\frac{1}{a^2}\frac{1}{1-a}+\cdots\right] = \frac{a}{1-a}\left[1+\frac{1}{a}+\frac{1}{a^2}+\cdots\right] = \boxed{\left(\frac{a}{1-a}\right)^2}$.

10. Define a monic irreducible polynomial with integral coefficients to be a polynomial with leading coefficient 1 that cannot be factored, and the prime factorization of a polynomial with leading coefficient 1 as the factorization into monic irreducible polynomials. How many not necessarily distinct monic irreducible polynomials are there in the prime factorization of $(x^8+x^4+1)(x^8+x+1)$ (for instance, $(x+1)^2$ has two prime factors)?

Solution: $x^8+x^4+1 = (x^8+2x^4+1)-x^4 = (x^4+1)^2-(x^2)^2 = (x^4-x^2+1)(x^4+x^2+1) = (x^4-x^2+1)(x^2+x+1)(x^2-x+1)$, and $x^8+x+1 = (x^2+x+1)(x^6-x^5+x^3-x^2+1)$. If an integer polynomial $f(x) = a_n x^n + \cdots + a_0 (\bmod p)$, where p does not divide a_n, has no zeros, then f

has no rational roots. Taking $p = 2$, we find $x^6 - x^5 + x^3 - x^2 + 1$ is irreducible. The prime factorization of our polynomial is thus $(x^4-x^2+1)(x^2-x+1)(x^2+x+1)^2(x^6-x^5+x^3-x^2+1)$, so the answer is $\boxed{5}$.

11. Define $a? = (a-1)/(a+1)$ for $a \neq -1$. Determine all real values N for which $(N?)? = \tan 15$.

Solution: Let $x = N?$. Then $(x-1)\cos 15 = (x+1)\sin 15$. Squaring and rearranging terms, and using the fact that $\cos^2 15 - \sin^2 15 = \cos 30 = \frac{\sqrt{3}}{2}$, we have $3x^2 - 4\sqrt{3}x + 3 = 0$. Solving, we find that $x = \sqrt{3}$ or $\frac{\sqrt{3}}{3}$. However, we may reject the second root because it yields a negative value for $(N?)?$. Therefore $x = \sqrt{3}$ and $N = \frac{1+x}{1-x} = \boxed{\frac{1+\sqrt{3}}{1-\sqrt{3}}} = \boxed{-2-\sqrt{3}}$.

12. All subscripts in this problem are to be considered modulo 6, that means for example that ω_7 is the same as ω_1. Let $\omega_1, \ldots \omega_6$ be circles of radius r, whose centers lie on a regular hexagon of side length 1. Let P_i be the intersection of ω_i and ω_{i+1} that lies further from the center of the hexagon, for $i = 1, \ldots 6$. Let Q_i, $i = 1 \ldots 6$, lie on ω_i such that Q_i, P_i, Q_{i+1} are colinear. Find the number of possible values of r.

Solution: Consider two consecutive circles ω_i and ω_{i+1}. Let Q_i, Q'_i be two points on ω_i and Q_{i+1}, Q'_{i+1} on ω_{i+1} such that Q_i, P_i and Q_{i+1} are colinear and also Q'_i, P_i and Q'_{i+1}. Then $Q_iQ'_i = 2\angle Q_iP_iQ'_i = 2\angle Q_{i+1}P_iQ'_{i+1} = \angle Q_{i+1}Q'_{i+1}$. Refer to the center of ω_i as O_i. The previous result shows that the lines O_iQ_i and $O_{i+1}Q_{i+1}$ meet at the same angle as the lines $O_iQ'_i$ and $O_{i+1}Q'_{i+1}$, call this angle ψ_i. ψ_i is a function solely of the circles ω_i and ω_{i+1} and the distance between them (we have just showed that any two points Q_i and Q'_i on ω_i give the same value of ψ_i, so ψ_i can't depend on this.) Now, the geometry of ω_i and ω_{i+1} is the same for every i, so ψ_i is simply a constant ψ which depends only on r. We know $6\psi = 0 \bmod 2\pi$ because $Q_7 = Q_1$.

We now compute ψ. It suffices to do the computaiton for some specific choice of Q_i. Take Q_i to be the intersection of O_iO_{i+1} and ω_i which is further from O_{i+1}. We are to compute the angle between O_iQ_i and $O_{i+1}Q_{i+1}$ which is the same as $\angle O_iO_{i+1}Q_{i+1}$. Note the triangle $\triangle O_iP_iO_{i+1}$ is isosceles, call the base angle ξ. We have $\angle O_iO_{i+1}Q_{i+1} = \angle O_iO_{i+1}P_i + \angle P_iO_{i+1}Q_{i+1} = \xi + (\pi - 2\angle O_{i+1}P_iQ_{i+1}) = \xi + (\pi - 2(\pi - \angle Q_iO_{i+1}P_i - \angle P_iQ_iO_{i+1})) = \xi - \pi + 2(\xi + (1/2)\angle P_iO_iO_{i+1}) = \xi - \pi + 2(\xi + (1/2)\xi) = 4\xi - \pi$.

So we get $6(4\xi - \pi) = 0 \bmod 2\pi$. Noting that ξ must be acute, $\xi = \pi/12, \pi/6, \pi/4, \pi/3$ or $5\pi/12$. r is uniquely determined as $(1/2)\sec\xi$ so there are $\boxed{5}$ possible values of r.