Harvard-MIT Math Tournament
March 17, 2002
Individual Subject Test: **Calculus**

1. Two circles have centers that are d units apart, and each has diameter $\sqrt{d}$. For any d, let $A(d)$ be the area of the smallest circle that contains both of these circles. Find $\lim_{d\to\infty} \frac{A(d)}{d^2}$.

Solution: This equals $\lim_{d\to\infty} \frac{\pi\left(\frac{d+\sqrt{d}}{2}\right)^2}{d^2} = \boxed{\frac{\pi}{4}}$.

2. Find $\lim_{h\to 0} \frac{x^2-(x+h)^2}{h}$.

Solution: This equals $\lim_{h\to 0} \frac{x^2-x^2-h^2-2hx}{h} = \lim_{h\to 0} -h-2x = \boxed{-2x}$. Alternate Solution: This is the definition of the derivative of $-x^2$ with respect to x, which is $-2x$.

3. We are given the values of the differentiable real functions f, g, h, as well as the derivatives of their pairwise products, at $x = 0$:

$$f(0) = 1;\ g(0) = 2;\ h(0) = 3;\ (gh)'(0) = 4;\ (hf)'(0) = 5;\ (fg)'(0) = 6.$$

Find the value of $(fgh)'(0)$.

Solution: $\boxed{16}$ By the product rule, $(fgh)' = f'gh + fg'h + fgh' = ((fg)'h + (gh)'f + (hf)'g)/2$. Evaluated at 0, this gives 16.

4. Find the area of the region in the first quadrant $x > 0, y > 0$ bounded above the graph of $y = \arcsin(x)$ and below the graph of the $y = \arccos(x)$.

Solution: We can integrate over y rather than x. In particular, the solution is $\int_0^{\pi/4} \sin y\ \mathrm{d}y + \int_{\pi/4}^{\pi/2} \cos y\ \mathrm{d}y = \left(1 - \frac{1}{\sqrt{2}}\right) 2 = \boxed{2-\sqrt{2}}$.

5. What is the minimum vertical distance between the graphs of $2 + \sin(x)$ and $\cos(x)$?

Solution: The derivative of $2 + \sin(x) - \cos(x)$ is $\cos x + \sin x$, which in the interval $0 \le x < 2\pi$ is zero at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. At $\frac{7\pi}{4}$, when $\sin(x)$ is negative and $\cos(x)$ is positive, the distance reaches its minimal value of $\boxed{2-\sqrt{2}}$.

6. Determine the positive value of a such that the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices $(0, 0), (a, 0), (0, a^2 + 1)$, and $(a, a^2 + 1)$.

Solution: $\boxed{\sqrt{3}}$ The area of the rectangle is $a^3 + a$. The portion under the parabola has area $\int_0^a x^2 + 1dx = a^3/3 + a$. Thus we wish to solve the equation $a^3 + a = 2(a^3/3 + a)$; dividing by a and rearranging gives $a^2/3 = 1$, so $a = \sqrt{3}$.

7. Denote by $\langle x\rangle$ the fractional part of the real number x (for instance, $\langle 3.2\rangle = 0.2$). A positive integer N is selected randomly from the set $\{1,2,3,\ldots,M\}$, with each integer having the same probability of being picked, and $\left\langle \frac{87}{303}N\right\rangle$ is calculated. This procedure is repeated M times and the average value $A(M)$ is obtained. What is $\lim_{M\to\infty} A(M)$?

Solution: This method of picking N is equivalent to uniformly randomly selecting a positive integer. Call this the average value of $\left\langle \frac{87}{303}N\right\rangle$ for N a positive integer. In lowest terms, $\frac{87}{303} = \frac{29}{101}$, so the answer is the same as the average value of $\frac{0}{101}, \frac{1}{101}, \ldots, \frac{100}{101}$, which is $\frac{1+2+\cdots+100}{101\cdot 101} = \frac{100\cdot 101/2}{101\cdot 101} = \boxed{\frac{50}{101}}$.

8. Evaluate $\displaystyle\int_0^{(\sqrt{2}-1)/2} \frac{dx}{(2x+1)\sqrt{x^2+x}}$.

Solution: Let $u = \sqrt{x^2+x}$. Then $du = \frac{2x+1}{2\sqrt{x^2+x}}\,dx$. So the integral becomes $2\int \frac{du}{(4x^2+4x+1)}$, or $2\int \frac{du}{4u^2+1}$. This is $\tan^{-1}(2u)$, yielding a final answer of $\tan^{-1}(2\sqrt{x^2+x}) + C$ for the indefinite integral. The definite integral becomes $\tan^{-1}(1) - \tan^{-1}(0) = \boxed{\frac{\pi}{4}}$.

9. Suppose f is a differentiable real function such that $f(x) + f'(x) \le 1$ for all x, and $f(0) = 0$. What is the largest possible value of $f(1)$? (Hint: consider the function $e^x f(x)$.)

Solution: $\boxed{1 - 1/e}$ Let $g(x) = e^x f(x)$; then $g'(x) = e^x(f(x) + f'(x)) \le e^x$. Integrating from 0 to 1, we have $g(1) - g(0) = \int_0^1 g'(x)dx \le \int_0^1 e^x dx = e - 1$. But $g(1) - g(0) = e\cdot f(1)$, so we get $f(1) \le (e-1)/e$. This maximum is attained if we actually have $g'(x) = e^x$ everywhere; this entails the requirement $f(x) + f'(x) = 1$, which is met by $f(x) = 1 - e^{-x}$.

10. A continuous real function f satisfies the identity $f(2x) = 3f(x)$ for all x. If $\int_0^1 f(x)\,dx = 1$, what is $\int_1^2 f(x)\,dx$?

Solution: $\boxed{5}$ Let $S = \int_1^2 f(x)\,dx$. By setting $u = 2x$, we see that $\int_{1/2}^1 f(x)\,dx = \int_{1/2}^1 f(2x)/3\ dx = \int_1^2 f(u)/6\ du = S/6$. Similarly, $\int_{1/4}^{1/2} f(x)\,dx = S/36$, and in general $\int_{1/2^n}^{1/2^{n-1}} f(x)\,dx = S/6^n$. Adding finitely many of these, we have $\int_{1/2^n}^1 f(x)\,dx = S/6 + S/36 + \cdots + S/6^n = S\cdot(1-1/6^n)/5$. Taking the limit as $n\to\infty$, we have $\int_0^1 f(x)\,dx = S/5$. Thus $S = 5$, the answer.