Team Event Solutions
HMMT 2002

Palindromes. A *palindrome* is a positive integer n not divisible by 10 such that if you write the decimal digits of n in reverse order, the number you get is n itself. For instance, the numbers 4 and 25752 are palindromes.

1. [15] Determine the number of palindromes that are less than 1000.

Solution. Every one-digit number (there are nine) is a palindrome. The two-digit palindromes have the form $\underline{a}\,\underline{a}$ for a nonzero digit a, so there are nine of them. A three-digit palindrome is $\underline{a}\,\underline{b}\,\underline{a}$ with a a nonzero digit and b any digit, so there are $9 \times 10 = 90$ of these. Thus the number of palindromes less than 1000 is $9 + 9 + 90 = \mathbf{108}$.

2. [30] Determine the number of four-digit integers n such that n and $2n$ are both palindromes.

Solution. Let $n = \underline{a}\,\underline{b}\,\underline{b}\,\underline{a}$. If $a, b \leq 4$ then there are no carries in the multiplication $n \times 2$, and $2n = \underline{(2a)}\,\underline{(2b)}\,\underline{(2b)}\,\underline{(2a)}$ is a palindrome. We shall show conversely that if n and $2n$ are palindromes, then necessarily $a, b \leq 4$. Hence the answer to the problem is $4 \times 5 = \mathbf{20}$ (because a cannot be zero).

If $a \geq 5$ then $2n$ is a five-digit number whose most significant digit is 1, but because $2n$ is even, its least significant digit is even, contradicting the assumption that $2n$ is a palindrome. Therefore $a \leq 4$. Consequently $2n$ is a four-digit number, and its tens and hundreds digits must be equal. Because $a \leq 4$, there is no carry out of the ones place in the multiplication $n \times 2$, and therefore the tens digit of $2n$ is the ones digit of $2b$. In particular, the tens digit of $2n$ is even. But if $b \geq 5$, the carry out of the tens place makes the hundreds digit of $2n$ *odd*, which is impossible. Hence $b \leq 4$ as well.

3. [40] Suppose that a positive integer n has the property that n, $2n$, $3n$, ..., $9n$ are all palindromes. Prove that the decimal digits of n are all zeros or ones.

Solution. First consider the ones digit a of n; we claim that $a = 1$. Certainly a cannot be even, for then $5n$ would be divisible by 10. If a is 5, 7, or 9, then $2n$ has an even ones digit, while its most significant digit is 1. If a is 3, then $4n$ has an even ones digit but most significant digit 1. Thus $a = 1$ is the only possibility. Moreover $9n$ has the same number of digits as n, for otherwise $9n$ would have most significant digit 1 but least significant digit 9, which is forbidden.

Now suppose n has at least one digit that is neither a zero nor a one. Let b be the leftmost (i.e., most significant) such digit, so that the left end of the decimal representation of n looks like

$$\underline{a_1} \cdots \underline{a_r}\,\underline{b} \cdots$$

for some $r \geq 1$ and digits $a_i \in \{0, 1\}$. When n is multiplied by 9, there will be a carry out of the column containing b. In particular, the r^{th} digit from the left in $9n$ will not be $9a_r$. But the right end of the decimal representation of n is

$$\cdots \underline{a_r} \cdots \underline{a_1}\,;$$

because each a_i is 0 or 1, there are no carries out of the first $r - 1$ columns, so the r^{th} digit from the right in $9n$ will be $9a_r$. Thus $9n$ is not a palindrome, a contradiction. This completes the proof.

Floor functions. The notation $\lfloor x \rfloor$ stands for the largest integer less than or equal to x.

4. [15] Let n be an integer. Prove that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor = n.$$

Solution. Suppose $n = 2m$ is even. Then $\lfloor n/2 \rfloor = \lfloor m \rfloor = m$ and $\lfloor (n+1)/2 \rfloor = \lfloor m + 1/2 \rfloor = m$, whose sum is $m+m = 2m = n$. Otherwise $n = 2m+1$ is odd. In this case $\lfloor n/2 \rfloor = \lfloor m + 1/2 \rfloor = m$ and $\lfloor (n+1)/2 \rfloor = \lfloor m+1 \rfloor = m+1$, whose sum is $m + (m+1) = 2m+1 = n$, as desired.

5. [20] Prove for integers n that

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Solution. Suppose $n = 2m$ is even; then $\lfloor n/2 \rfloor = \lfloor m \rfloor = m$ and $\lfloor (n+1)/2 \rfloor = \lfloor m + 1/2 \rfloor = m$, whose product is $m^2 = \lfloor m^2 \rfloor = \lfloor (2m)^2/4 \rfloor$. Otherwise $n = 2m+1$ is odd, so that $\lfloor n/2 \rfloor = \lfloor m + 1/2 \rfloor = m$ and $\lfloor (n+1)/2 \rfloor = \lfloor m+1 \rfloor = m+1$, whose product is $m^2 + m$. On the other side, we find that

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{4m^2+4m+1}{4} \right\rfloor = \left\lfloor m^2 + m + \frac{1}{4} \right\rfloor = m^2 + m,$$

as desired.

In problems 6–7 you may use without proof the known summations

$$\sum_{n=1}^{L} n = n(n+1)/2 \quad \text{and} \quad \sum_{n=1}^{L} n^3 = n^2(n+1)^2/4 \quad \text{for positive integers } L.$$

6. [20] For positive integers L, let $S_L = \sum_{n=1}^{L} \lfloor n/2 \rfloor$. Determine all L for which S_L is a square number.

Solution. We distinguish two cases depending on the parity of L. Suppose first that $L = 2k-1$ is odd, where $k \geq 1$. Then

$$S_L = \sum_{1 \leq n \leq 2k-1} \left\lfloor \frac{n}{2} \right\rfloor = 2 \sum_{0 \leq m < k} m = 2 \cdot \frac{k(k-1)}{2} = k(k-1).$$

If $k = 1$, this is the square number 0. If $k > 1$ then $(k-1)^2 < k(k-1) < k^2$, so $k(k-1)$ is not square. Now suppose $L = 2k$ is even, where $k \geq 1$. Then $S_L = S_{L-1} + k = k^2$ is always square. Hence S_L is square exactly when $L =$ **1 or** L **is even**.

7. [45] Let $T_L = \sum_{n=1}^{L} \lfloor n^3/9 \rfloor$ for positive integers L. Determine all L for which T_L is a square number.

Solution. Since T_L is square if and only if $9T_L$ is square, we may consider $9T_L$ instead of T_L.

It is well known that n^3 is congruent to 0, 1, or 8 modulo 9 according as n is congruent to 0, 1, or 2 modulo 3. (Proof: $(3m+k)^3 = 27m^3 + 3(9m^2)k + 3(3m)k^2 + k^3 \equiv k^3 \pmod 9$.) Therefore

$n^3 - 9\lfloor n^3/9 \rfloor$ is 0, 1, or 8 according as n is congruent to 0, 1, or 2 modulo 3. We find therefore that

$$\begin{aligned} 9T_L &= \sum_{1\le n\le L} 9\left\lfloor \frac{n^3}{9} \right\rfloor \\ &= \sum_{1\le n\le L} n^3 - \#\{1 \le n \le L : n \equiv 1 \pmod 3\} - 8\#\{1 \le n \le L : n \equiv 2 \pmod 3\} \\ &= \left(\frac{1}{2}L(L+1)\right)^2 - \left\lfloor \frac{L+2}{3} \right\rfloor - 8\left\lfloor \frac{L+1}{3} \right\rfloor. \end{aligned}$$

Clearly $9T_L < \big(L(L+1)/2\big)^2$ for $L \ge 1$. We shall prove that $9T_L > \big(L(L+1)/2 - 1\big)^2$ for $L \ge 4$, whence $9T_L$ is not square for $L \ge 4$. Because

$$\big(L(L+1)/2 - 1\big)^2 = \big(L(L+1)/2\big)^2 - L(L+1) + 1,$$

we need only show that

$$\left\lfloor \frac{L+2}{3} \right\rfloor + 8\left\lfloor \frac{L+1}{3} \right\rfloor \le L^2 + L - 2.$$

But the left-hand side of this is bounded above by $3L+10/3$, and the inequality $3L+10/3 \le L^2+L-2$ means exactly $L^2 - 2L - 16/3 \ge 0$ or $(L-1)^2 \ge 19/3$, which is true for $L \ge 4$, as desired.

Hence T_L is not square for $L \ge 4$. By direct computation we find $T_1 = T_2 = 0$ and $T_3 = 3$, so T_L is square only for $L \in \{\mathbf{1}, \mathbf{2}\}$.

Luck of the dice. Problems 8–12 concern a two-player game played on a board consisting of fourteen spaces in a row. The leftmost space is labeled *START*, and the rightmost space is labeled *END*. Each of the twelve other squares, which we number 1 through 12 from left to right, may be blank or may be labeled with an arrow pointing to the right. The term *blank square* will refer to one of these twelve squares that is not labeled with an arrow. The set of blank squares on the board will be called a *board configuration*; the board below uses the configuration $\{1, 2, 3, 4, 7, 8, 10, 11, 12\}$.

START					$\Rightarrow$	$\Rightarrow$			$\Rightarrow$				*END*
	1	2	3	4	5	6	7	8	9	10	11	12	

For $i \in \{1, 2\}$, player i has a die that produces each integer from 1 to s_i with probability $1/s_i$. Here s_1 and s_2 are positive integers fixed before the game begins. The game rules are as follows:

1. The players take turns alternately, and player 1 takes the first turn.
2. On each of his turns, player i rolls his die and moves his piece to the right by the number of squares that he rolled. If his move ends on a square marked with an arrow, he moves his piece forward another s_i squares. If that move ends on an arrow, he moves another s_i squares, repeating until his piece comes to rest on a square without an arrow.
3. If a player's move would take him past the *END* square, instead he lands on the *END* square.
4. Whichever player reaches the *END* square first wins.

As an example, suppose that $s_1 = 3$ and the first player is on square 4 in the sample board shown above. If the first player rolls a 2, he moves to square 6, then to square 9, finally coming to rest on square 12. If the second player does not reach the *END* square on her next turn, the first player will necessarily win on his next turn, as he must roll at least a 1.

8. [35] In this problem only, assume that $s_1 = 4$ and that exactly one board square, say square number n, is marked with an arrow. Determine all choices of n that maximize the average distance in squares the first player will travel in his first two turns.

Solution. Because expectation is linear, the average distance the first player travels in his first two turns is the average sum of two rolls of his die (which does not depend on the board configuration) plus four times the probability that he lands on the arrow on one of his first two turns. Thus we just need to maximize the probability that player 1 lands on the arrow in his first two turns. If $n \geq 5$, player 1 cannot land on the arrow in his first turn, so he encounters the arrow with probability at most $1/4$. If instead $n \leq 4$, player 1 has a $1/4$ chance of landing on the arrow on his first turn. If he misses, then he has a $1/4$ chance of hitting the arrow on his second turn provided that he is not beyond square n already. The chance that player 1's first roll left him on square $n-1$ or farther left is $(n-1)/4$. Hence his probability of benefiting from the arrow in his first two turns is $1/4 + (1/4)(n-1)/4$, which is maximized for $n = 4$, where it is greater than the value of $1/4$ that we get from $n \geq 5$. Hence the answer is $n = \mathbf{4}$.

9. [30] In this problem suppose that $s_1 = s_2$. Prove that for each board configuration, the first player wins with probability strictly greater than $\frac{1}{2}$.

Solution. Let σ_1 and σ_2 denote the sequence of the next twelve die rolls that players 1 and 2 respectively will make. The outcome of the game is completely determined by the σ_i. Now player 1 wins in all cases in which $\sigma_1 = \sigma_2$, for then each of player 2's moves bring her piece to a square already occupied by player 1's piece. It is sufficient, therefore, to show that player 1 wins at least half the cases in which $\sigma_1 \neq \sigma_2$. But all these cases can be partitioned into disjoint pairs

$$\{(\sigma_1, \sigma_2), (\sigma_2, \sigma_1)\},$$

and player 1 wins in at least one case in each pair. For if player 2 wins in the case (σ_1, σ_2), say on her n^{th} turn, the first n elements of σ_1 do not take player 1 beyond space 12, while the first n elements of σ_2 must take player 2 beyond space 12. Clearly, then, player 1 wins (σ_2, σ_1) on his n^{th} turn.

10. [30] Exhibit a configuration of the board and a choice of s_1 and s_2 so that $s_1 > s_2$, yet the *second* player wins with probability strictly greater than $\frac{1}{2}$.

Solution. Let $s_1 = 3$ and $s_2 = 2$ and place an arrow on all the even-numbered squares. In this configuration, player 1 can move at most six squares in a turn: up to three from his roll and an additional three if his roll landed him on an arrow. Hence player 1 cannot win on his first or second turn. Player 2, however, wins immediately if she ever lands on an arrow. Thus player 2 has probability $1/2$ of winning on her first turn, and failing that, she has probability $1/2$ of winning on her second turn. Hence player 2 wins with probability at least $1/2 + (1/2)(1/2) = 3/4$.

11. [55] In this problem assume $s_1 = 3$ and $s_2 = 2$. Determine, with proof, the nonnegative integer k with the following property:

1. For every board configuration with strictly fewer than k blank squares, the first player wins with probability strictly greater than $\frac{1}{2}$; but

2. there exists a board configuration with exactly k blank squares for which the second player wins with probability strictly greater than $\frac{1}{2}$.

Solution. The answer is $k = \mathbf{3}$. Consider the configuration whose blank squares are 2, 6, and 10. Because these numbers represent all congruence classes modulo 3, player 1 cannot win on his first turn: he will come to rest on one of the blank squares. But player 2 will win on her first turn if she rolls a 1, for 2, 6, and 10 are all even. Thus player 2 wins on her first turn with probability 1/2. Failing this, player 1 may fail to win on his second turn, for instance, if he rolled a 2 previously and now rolls a 1, ending up on square 6. Then player 2 will again have probability 1/2 of winning on her next turn. Thus player 2 wins the game with probability exceeding 1/2.

We must now prove that all configurations with fewer than three blanks favor player 1. If the numbers of the blank squares represent at most one residue class modulo 3, then clearly player 1 wins on his first turn with probability at least 2/3. This disposes of the cases of no blanks, just one blank, and two blanks that are congruent modulo 3. In the remaining case, there are two blank squares whose indices are incongruent modulo 3. Then player 1 wins on his first turn with probability only 1/3. If he does not win immediately, player 2 wins on her first turn with probability at most 1/2, for there is a blank in at least one congruence class modulo 2. If player 2 does not win on her first turn, then player 1 wins on his second turn with probability at least 2/3, for there is only one blank square in front of him now. Thus player 1 wins the game with probability at least $1/3 + (2/3)(1/2)(2/3) = 5/9 > 1/2$, as desired.

12. [65] Now suppose that before the game begins, the players choose the initial game state as follows:

1. The first player chooses s_1 subject to the constraint that $2 \le s_1 \le 5$; then
2. the second player chooses s_2 subject to the constraint that $2 \le s_2 \le 5$ and then specifies the board configuration.

Prove that the second player can always make her decisions so that she will win the game with probability strictly greater than $\frac{1}{2}$.

Solution. If $s_1 \in \{3, 5\}$, take $s_2 = 2$ and put arrows on the even-numbered squares. Player 1 cannot win on his first turn because he can move at most $2s_1$ spaces in a turn. Player 2 wins on her first turn with probability 1/2. Failing that, player 1 might fail to win on his second turn, and player 2 will again have probability 1/2 of winning on her second turn, so her probability of winning the game is certainly greater than 1/2.

If $s_1 = 4$, take $s_2 = 3$ and leave blank only squares 1, 4, 7, and 10. These occupy all congruence classes modulo 4, so player 1 cannot win on his first turn. But the blank squares lie in the same congruence class modulo 3, so player 2 then wins on her first turn with probability 2/3.

Finally, if $s_1 = 2$, take $s_2 = 5$ and leave all squares blank. Then player 2 moves 3 squares in a turn on average, hence covers 15 squares on average in her first five turns. Moreover, the distribution of player 2's distance traveled in five turns is symmetric about 15. Thus player 2 has probability greater than 1/2 of reaching *END* by the end of her fifth turn. Player 1, on the other hand, cannot win in five turns because he can move at most 10 squares in those turns.