Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Algebra Subject Test — Solutions

1. Find the smallest value of x such that $a \geq 14\sqrt{a} - x$ for all nonnegative a.

   Solution: $\boxed{49}$

   We want to find the smallest value of x such that $x \geq 14 sqrta - a$ for all a. This is just the maximum possible value of $14\sqrt{a} - a = 49 - (\sqrt{a} - 7)^2$, which is clearly 49, achieved when $a = 49$.

2. Compute $\frac{\tan^2(20°) - \sin^2(20°)}{\tan^2(20°)\sin^2(20°)}$.

   Solution: $\boxed{1}$

   If we multiply top and bottom by $\cos^2(20°)$, the numerator becomes $\sin^2(20°) \cdot (1 - \cos^2 20°) = \sin^4(20°)$, while the denominator becomes $\sin^4(20°)$ also. So they are equal, and the ratio is 1.

3. Find the smallest n such that $n!$ ends in 290 zeroes.

   Solution: $\boxed{1170}$

   Each 0 represents a factor of $10 = 2 \cdot 5$. Thus, we wish to find the smallest factorial that contains at least 290 2's and 290 5's in its prime factorization. Let this number be $n!$, so the factorization of $n!$ contains 2 to the power p and 5 to the power q, where

   $$p = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \cdots \text{ and } q = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots$$

   (this takes into account one factor for each single multiple of 2 or 5 that is $\leq n$, an additional factor for each multiple of 2^2 or 5^2, and so on). Naturally, $p \geq q$ because 2 is smaller than 5. Thus, we want to bring q as low to 290 as possible. If $q = \lfloor \frac{n}{5} \rfloor + \lfloor \frac{n}{5^2} \rfloor + \lfloor \frac{n}{5^3} \rfloor + \cdots$, we form a rough geometric sequence (by taking away the floor function) whose sum is represented by $290 \approx \frac{n/5}{1-1/5}$. Hence we estimate $n = 1160$, and this gives us $q = 288$. Adding 10 to the value of n gives the necessary two additional factors of 5, and so the answer is 1170.

4. Simplify: $2\sqrt{1.5 + \sqrt{2}} - (1.5 + \sqrt{2})$.

   Solution: $\boxed{1/2}$

   The given expression equals $\sqrt{6 + 4\sqrt{2}} - (1.5 + \sqrt{2}) = \sqrt{6 + 2\sqrt{8}} - (1.5 + \sqrt{2})$. But on inspection, we see that $(\sqrt{2} + \sqrt{4})^2 = 6 + 2\sqrt{8}$, so the answer is $(\sqrt{2} + \sqrt{4}) - (1.5 + \sqrt{2}) = 2 - 3/2 = 1/2$.

5. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that n_1 of the given numbers are equal to 1, n_2 of them are equal to 2, ..., n_{2003} of them are equal to 2003. Find the largest possible value of

   $$n_2 + 2n_3 + 3n_4 + \cdots + 2002n_{2003}.$$

Solution: $\boxed{2002}$

The sum of all the numbers is $n_1 + 2n_2 + \cdots + 2003n_{2003}$, while the number of numbers is $n_1 + n_2 + \cdots + n_{2003}$. Hence, the desired quantity equals

$$(n_1 + 2n_2 + \cdots + 2003n_{2003}) - (n_1 + n_2 + \cdots + n_{2003})$$
$$= (\text{sum of the numbers}) - (\text{number of numbers})$$
$$= 2003 - (\text{number of numbers}),$$

which is maximized when the number of numbers is minimized. Hence, we should have just one number, equal to 2003, and then the specified sum is $2003 - 1 = 2002$.

Comment: On the day of the contest, a protest was lodged (successfully) on the grounds that the use of the words "several" and "their" in the problem statement implies there must be at least 2 numbers. Then the answer is 2001, and this maximum is achieved by any two numbers whose sum is 2003.me way.)

6. Let $a_1 = 1$, and let $a_n = \lfloor n^3/a_{n-1} \rfloor$ for $n > 1$. Determine the value of a_{999}.

   Solution: $\boxed{999}$

   We claim that for any odd n, $a_n = n$. The proof is by induction. To get the base cases $n = 1, 3$, we compute $a_1 = 1$, $a_2 = \lfloor 2^3/1 \rfloor = 8$, $a_3 = \lfloor 3^3/8 \rfloor = 3$. And if the claim holds for odd $n \geq 3$, then $a_{n+1} = \lfloor (n+1)^3/n \rfloor = n^2 + 3n + 3$, so $a_{n+2} = \lfloor (n+2)^3/(n^2+3n+3) \rfloor = \lfloor (n^3+6n^2+12n+8)/(n^2+3n+2) \rfloor = \lfloor n+2+\frac{n^2+3n+2}{n^2+3n+3} \rfloor = n+2$. So the claim holds, and in particular, $a_{999} = 999$.

7. Let a, b, c be the three roots of $p(x) = x^3 + x^2 - 333x - 1001$. Find $a^3 + b^3 + c^3$.

   Solution: $\boxed{2003}$

   We know that $x^3 + x^2 - 333x - 1001 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$. Also, $(a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 3abc = a^3 + b^3 + c^3$. Thus, $a^3 + b^3 + c^3 = (-1)^3 - 3(-1)(-333) + 3 \cdot 1001 = 2003$.

8. Find the value of $\frac{1}{3^2+1} + \frac{1}{4^2+2} + \frac{1}{5^2+3} + \cdots$.

   Solution: $\boxed{13/36}$

   Each term takes the form

   $$\frac{1}{n^2 + (n-2)} = \frac{1}{(n+2) \cdot (n-1)}.$$

   Using the method of partial fractions, we can write (for some constants A, B)

   $$\frac{1}{(n+2) \cdot (n-1)} = \frac{A}{(n+2)} + \frac{B}{(n-1)}$$
   $$\Rightarrow 1 = A \cdot (n-1) + B \cdot (n+2)$$

   Setting $n = 1$ we get $B = \frac{1}{3}$, and similarly with $n = -2$ we get $A = -\frac{1}{3}$. Hence the sum becomes

   $$\frac{1}{3} \cdot \left[\left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \cdots \right].$$

   Thus, it telescopes, and the only terms that do not cancel produce a sum of $\frac{1}{3} \cdot (\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) = \frac{13}{36}$.

9. For how many integers n, for $1 \le n \le 1000$, is the number $\frac{1}{2}\binom{2n}{n}$ even?

Solution: $\boxed{990}$

In fact, the expression $\binom{2n}{n}$ is always even, and it is not a multiple of four if and only if n is a power of 2, and there are 10 powers of 2 between 1 and 1000.

Let $f(N)$ denote the number of factors of 2 in N. Thus,

$$f(n!) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \cdots = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor .$$

Also, it is clear that $f(ab) = f(a) + f(b)$ and $f(\frac{a}{b}) = f(a) - f(b)$ for integers a, b. Now for any positive integer n, let m be the integer such that $2^m \le n < 2^{m+1}$. Then

$$\begin{aligned} f\left(\binom{2n}{n}\right) = f\left(\frac{(2n)!}{n!n!}\right) &= \sum_{k=1}^{\infty} \left\lfloor \frac{2n}{2^k} \right\rfloor - 2\left(\sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor\right) \\ &= \sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^{k-1}} \right\rfloor - 2\left(\sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor\right) \\ &= \lfloor n \rfloor - \left(\sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor\right) \\ &= n - \left(\sum_{k=1}^{m} \left\lfloor \frac{n}{2^k} \right\rfloor\right) \\ &\ge n - \left(\sum_{k=1}^{m} \frac{n}{2^k}\right) \\ &= n - n\left(\frac{2^m - 1}{2^m}\right) = \frac{n}{2^m} \ge 1. \end{aligned}$$

Both equalities hold when $n = 2^m$, and otherwise, $f(\binom{2n}{n}) > 1$.

10. Suppose $P(x)$ is a polynomial such that $P(1) = 1$ and

$$\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$$

for all real x for which both sides are defined. Find $P(-1)$.

Solution: $\boxed{-5/21}$

Cross-multiplying gives $(x+7)P(2x) = 8xP(x+1)$. If P has degree n and leading coefficient c, then the leading coefficients of the two sides are $2^n c$ and $8c$, so $n = 3$. Now $x = 0$ is a root of the right-hand side, so it's a root of the left-hand side, so that $P(x) = xQ(x)$ for some polynomial $Q \Rightarrow 2x(x+7)Q(2x) = 8x(x+1)Q(x+1)$ or $(x+7)Q(2x) = 4(x+1)Q(x+1)$. Similarly, we see that $x = -1$ is a root of the left-hand side, giving $Q(x) = (x+2)R(x)$ for some polynomial $R \Rightarrow 2(x+1)(x+7)R(2x) = 4(x+1)(x+3)R(x+1)$, or $(x+7)R(2x) = 2(x+3)R(x+1)$. Now $x = -3$ is a root of the left-hand side, so $R(x) = (x+6)S(x)$ for some polynomial S.

At this point, $P(x) = x(x+2)(x+6)S(x)$, but P has degree 3, so S must be a constant. Since $P(1) = 1$, we get $S = 1/21$, and then $P(-1) = (-1)(1)(5)/21 = -5/21$.