Harvard-MIT Mathematics Tournament

March 15, 2003

Team Round — Solutions

Completions and Configurations

Given a set A and a nonnegative integer k, the *k-completion* of A is the collection of all k-element subsets of A, and a *k-configuration* of A is any subset of the k-completion of A (including the empty set and the entire k-completion). For instance, the 2-completion of $A = \{1, 2, 3\}$ is $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and the 2-configurations of A are

$$\begin{array}{cc} \{\} & \{\{1,2\}\} \\ \{\{1,3\}\} & \{\{2,3\}\} \\ \{\{1,2\},\{1,3\}\} & \{\{1,2\},\{2,3\}\} \\ \{\{1,3\},\{2,3\}\} & \{\{1,2\},\{1,3\},\{2,3\}\} \end{array}$$

The *order* of an element a of A with respect to a given k-configuration of A is the number of subsets in the k-configuration that contain a. A k-configuration of a set A is *consistent* if the order of every element of A is the same, and the *order* of a consistent k-configuration is this common value.

1. (a) How many k-configurations are there of a set that has n elements?

   Solution: An n-element set has $\binom{n}{k}$ subsets of size k, and we can construct a k-configuration by independently choosing, for each subset, whether or not to include it, so there are $2^{\binom{n}{k}}$ k-configurations.

   (b) How many k-configurations that have m elements are there of a set that has n elements?

   Solution: Again, an n-element set has $\binom{n}{k}$ subsets of size k, so there are $\binom{\binom{n}{k}}{m}$ k-configurations with m elements.

2. Suppose A is a set with n elements, and k is a divisor of n. Find the number of consistent k-configurations of A of order 1.

   Solution: Given such a k-configuration, we can write out all the elements of one of the k-element subsets, then all the elements of another subset, and so forth, eventually obtaining an ordering of all n elements of A. Conversely, given any ordering of the elements of A, we can construct a consistent k-configuration of order 1 from it by grouping together the first k elements, then the next k elements, and so forth. In fact, each consistent k-configuration of order 1 corresponds to $(n/k)!(k!)^{n/k}$ different such orderings, since the elements of A within each of the n/k k-element subsets can be ordered in $k!$ ways, and the various subsets can also be ordered with respect to each other in $(n/k)!$ different ways. Thus, since there are $n!$ orderings of the elements of A, we get $\frac{n!}{(n/k)!(k!)^{n/k}}$ different consistent k-configurations of order 1.

3. (a) Let $A_n = \{a_1, a_2, a_3, \ldots, a_n, b\}$, for $n \geq 3$, and let C_n be the 2-configuration consisting of $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq n-1$, $\{a_1, a_n\}$, and $\{a_i, b\}$ for $1 \leq i \leq n$. Let $S_e(n)$ be the number of subsets of C_n that are consistent of order e. Find $S_e(101)$ for $e = 1, 2$, and 3.

Solution: For convenience, we assume the a_i are indexed modulo 101, so that $a_{i+1} = a_1$ when $a_i = a_{101}$.

In any consistent subset of C_{101} of order 1, b must be paired with exactly one a_i, say a_1. Then, a_2 cannot be paired with a_1, so it must be paired with a_3, and likewise we find we use the pairs $\{a_4, a_5\}, \{a_6, a_7\}, \ldots, \{a_{100}, a_{101}\}$ — and this does give us a consistent subset of order 1. Similarly, pairing b with any other a_i would give us a unique extension to a consistent configuration of order 1. Thus, we have one such 2-configuration for each i, giving $S_1(101) = 101$ altogether.

In a consistent subset of order 2, b must be paired with two other elements. Suppose one of them is a_i. Then a_i is also paired with either a_{i-1} or a_{i+1}, say a_{i+1}. But then a_{i-1} needs to be paired up with two other elements, and a_i is not available, so it must be paired with a_{i-2} and b. Now b has its two pairs determined, so nothing else can be paired with b. Thus, for $j \neq i-1, i$, we have that a_j must be paired with a_{j-1} and a_{j+1}. So our subset must be of the form

$$\{\{b, a_i\}, \{a_i, a_{i+1}\}, \{a_{i+1}, a_{i+2}\}, \ldots, \{a_{101}, a_1\}, \ldots, \{a_{i-2}, a_{i-1}\}, \{a_{i-1}, b\}\}$$

for some i. On the other hand, for any $i = 1, \ldots, 101$, this gives a subset meeting our requirements. So, we have 101 possibilities, and $S_2(101) = 101$.

Finally, in a consistent subset of order 3, each a_i must be paired with a_{i-1}, a_{i+1}, and b. But then b occurs in 101 pairs, not just 3, so we have a contradiction. Thus, no such subset exists, so $S_3(101) = 0$.

(b) Let $A = \{V, W, X, Y, Z, v, w, x, y, z\}$. Find the number of subsets of the 2-configuration

$$\{\ \{V, W\}, \{W, X\}, \{X, Y\}, \{Y, Z\}, \{Z, V\},\ \{v, x\}, \{v, y\}, \{w, y\}, \{w, z\}, \{x, z\},$$
$$\{V, v\}, \{W, w\}, \{X, x\}, \{Y, y\}, \{Z, z\}\ \}$$

that are consistent of order 1.

Solution: No more than two of the pairs $\{v, x\}, \{v, y\}, \{w, y\}, \{w, z\}, \{x, z\}$ may be included in a 2-configuration of order 1, since otherwise at least one of v, w, x, y, z would occur more than once. If exactly one is included, say $\{v, x\}$, then w, y, z must be paired with W, Y, Z, respectively, and then V and X cannot be paired. So either none or exactly two of the five pairs above must be used. If none, then v, w, x, y, z must be paired with V, W, X, Y, Z, respectively, and we have 1 2-configuration arising in this manner. If exactly two are used, we can check that there are 5 ways to do this without duplicating an element:

$$\{v, x\}, \{w, y\} \qquad \{v, x\}, \{w, z\} \qquad \{v, y\}, \{w, z\} \qquad \{v, y\}, \{x, z\} \qquad \{w, y\}, \{x, z\}$$

In each case, it is straightforward to check that there is a unique way of pairing up the remaining elements of A. So we get 5 2-configurations in this way, and the total is 6.

(c) Let $A = \{a_1, b_1, a_2, b_2, \ldots, a_{10}, b_{10}\}$, and consider the 2-configuration C consisting of $\{a_i, b_i\}$ for all $1 \leq i \leq 10$, $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq 9$, and $\{b_i, b_{i+1}\}$ for all $1 \leq i \leq 9$. Find the number of subsets of C that are consistent of order 1.

Solution: Let $A_n = \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$ for $n \geq 1$, and consider the 2-configuration C_n consisting of $\{a_i, b_i\}$ for all $1 \leq i \leq n$, $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq n-1$, and $\{b_i, b_{i+1}\}$

for all $1 \leq i \leq n-1$. Let N_n be the number of subsets of C_n that are consistent of order 1 (call these "matchings" of C_n). Consider any matching of C_{n+2}. Either a_{n+2} is paired with b_{n+2}, in which case the remaining elements of our matching form a matching of C_{n+1}; or a_{n+2} is paired with a_{n+1}, in which case b_{n+2} must be paired with b_{n+1}, and the remaining elements form a matching of C_n. It follows that $N_{n+2} = N_{n+1} + N_n$. By direct calculation, $N_1 = 1$ and $N_2 = 2$, and now computing successive values of N_n using the recurrence yields $N_{10} = 89$.

Define a k-configuration of A to be *m-separable* if we can label each element of A with an integer from 1 to m (inclusive) so that there is no element E of the k-configuration all of whose elements are assigned the same integer. If C is any subset of A, then C is *m-separable* if we can assign an integer from 1 to m to each element of C so that there is no element E of the k-configuration such that $E \subseteq C$ and all elements of E are assigned the same integer.

4. (a) Suppose A has n elements, where $n \geq 2$, and C is a 2-configuration of A that is not m-separable for any $m < n$. What is (in terms of n) the smallest number of elements that C can have?

Solution: We claim that every pair of elements of A must belong to C, so that the answer is $\binom{n}{2}$. Indeed, if $a, b \in A$ and $\{a, b\}$ is not in the 2-configuration, then we can assign the other elements of A the numbers $1, 2, \ldots, n-2$ and assign a and b both the number $n-1$, so that C is $(n-1)$-separable. On the other hand, if every pair of elements of A is in the configuration, then A cannot be m-separable for $m < n$, since this would require assigning the same number to at least two elements, and then we would have a pair whose elements have the same number.

(b) Show that every 3-configuration of an n-element set A is m-separable for every integer $m \geq n/2$.

Solution: We can successively label the elements of A with the numbers $1, 1, 2, 2, 3, 3, \ldots, \lceil n/2 \rceil$. Then surely no 3-element subset can have all its elements labeled with the same number, since no label is assigned to more than two elements. Thus, when $m \geq n/2 \Rightarrow m \geq \lceil n/2 \rceil$, this labeling shows that any 3-configuration is m-separable.

(c) Fix $k \geq 2$, and suppose A has k^2 elements. Show that any k-configuration of A with fewer than $\binom{k^2-1}{k-1}$ elements is k-separable.

Solution: The argument is similar to that used in problem 2. Suppose the configuration is not k-separable. Consider all possible orderings of the k^2 elements of A. For each ordering, assign the first k elements the number 1, the next k elements the number 2, and so forth. By assumption, for each such assignment, there exists some element of the k-configuration whose elements are all assigned the same number. Now consider any given element E of the k-configuration. For each i, we count the orderings in which all k elements of E receive the number i: there are $k!$ possible orderings for the elements of E, and there are $(k^2-k)!$ possible orderings for the remaining elements of A. Altogether, this gives $k \cdot k! \cdot (k^2-k)!$ orderings in which the elements of E all receive the same label. So if, in each of the $(k^2)!$ orderings of the elements of A, there is some E all of whose members receive the same label, then there must be at least

$$\frac{(k^2)!}{k \cdot k! \cdot (k^2-k)!} = \frac{(k^2-1)!}{(k-1)!(k^2-k)!} = \binom{k^2-1}{k-1}$$

elements E of the k-configuration. Hence, if there are fewer elements, the k-configuration is k-separable, as desired.

5. Let $B_k(n)$ be the largest possible number of elements in a 2-separable k-configuration of a set with $2n$ elements ($2 \le k \le n$). Find a closed-form expression (i.e. an expression not involving any sums or products with a variable number of terms) for $B_k(n)$.

   Solution: First, a lemma: For any a with $0 \le a \le 2n$, $\binom{a}{k} + \binom{2n-a}{k} \ge 2\binom{n}{k}$. (By convention, we set $\binom{a}{k} = 0$ when $a < k$.) Proof: We may assume $a \ge n$, since otherwise we can replace a with $2n - a$. Now we prove the result by induction on a. For the base case, if $a = n$, then the lemma states that $2\binom{n}{k} \ge 2\binom{n}{k}$, which is trivial. If the lemma holds for some $a > 0$, then by the familiar identity,

$$\begin{aligned} & \left[\binom{a+1}{k} + \binom{2n-a-1}{k}\right] - \left[\binom{a}{k} + \binom{2n-a}{k}\right] \\ = & \left[\binom{a+1}{k} - \binom{a}{k}\right] - \left[\binom{2n-a}{k} + \binom{2n-a-1}{k}\right] \\ = & \binom{a}{k-1} - \binom{2n-a-1}{k-1} > 0 \end{aligned}$$

   (since $a > 2n - a - 1$), so $\binom{a+1}{k} + \binom{2n-a-1}{k} > \binom{a}{k} + \binom{2n-a}{k} \ge 2\binom{n}{k}$, giving the induction step. The lemma follows.

   Now suppose that the elements of A are labeled such that a elements of the set A receive the number 1 and $2n - a$ elements receive the number 2. Then the k-configuration can include all k-element subsets of A except those contained among the a elements numbered 1 or the $2n - a$ elements numbered 2. Thus, we have at most $\binom{2n}{k} - \binom{a}{k} - \binom{2n-a}{k}$ elements in the k-configuration, and by the lemma, this is at most

$$\binom{2n}{k} - 2\binom{n}{k}.$$

   On the other hand, we can achieve $\binom{2n}{k} - 2\binom{n}{k}$ via the recipe above — take all the k-element subsets of A, except those contained entirely within the first n elements or entirely within the last n elements. Then, labeling the first n elements with the number 1 and the last n elements with the number 2 shows that the configuration is 2-separable. So, $B_k(n) = \binom{2n}{k} - 2\binom{n}{k}$.

6. Prove that any 2-configuration containing e elements is m-separable for some $m \le \frac{1}{2} + \sqrt{2e + \frac{1}{4}}$.

   Solution: Suppose m is the minimum integer for which the given configuration C on set A is m-separable, and fix a corresponding labeling of the elements of A. Let A_i be the set of all elements with the label i. Then, for any i, j with $1 \le i < j \le m$, there must exist $a_i \in A_i, a_j \in A_j$ with $\{a_i, a_j\} \in C$, since otherwise the elements of A_j could have been reassigned the label i, decreasing the number of distinct labels necessary and thus contradicting the minimality of m. We thus get at least $\binom{m}{2}$ different elements of C. Therefore, $e \ge \binom{m}{2} = m(m-1)/2 = [(m - \frac{1}{2})^2 - \frac{1}{4}]/2$, and solving for m gives the desired result.

A *cell* of a 2-configuration of a set A is a nonempty subset C of A such that

i. for any two distinct elements a, b of C, there exists a sequence $c_0, c_1, \ldots, c_n$ of elements of A with $c_0 = a, c_n = b$, and such that $\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}$ are all elements of the 2-configuration, and

ii. if a is an element of C and b is an element of A but not of C, there does NOT exist a sequence $c_0, c_1, \ldots, c_n$ of elements of A with $c_0 = a, c_n = b$, and such that $\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}$ are all elements of the 2-configuration.

Also, we define a 2-configuration of A to be *barren* if there is no subset $\{a_0, a_1, \ldots, a_n\}$ of A, with $n \geq 2$, such that $\{a_0, a_1\}, \{a_1, a_2\}, \ldots, \{a_{n-1}, a_n\}$ and $\{a_n, a_0\}$ are all elements of the 2-configuration.

7. Show that, given any 2-configuration of a set A, every element of A belongs to exactly one cell.

   Solution: First, given a, let C_a be the set of all $b \in A$ for which there exists a sequence $a = c_0, c_1, \ldots, c_n = b$ as in the definition of a cell. Certainly $a \in C_a$ (take $n = 0$); we claim that C_a is a cell. If $b, b' \in C_a$, then there exist sequences $a = c_0, c_1, \ldots, c_n = b$ and $a = c'_0, c'_1, \ldots, c'_m = b'$, so the sequence $b = c_n, c_{n-1}, \ldots, c_1, c_0, c'_1, \ldots, c'_m = b'$ shows that the first condition is met. For the second, suppose that there does exist a sequence $b = c_0, c_1, \ldots, c_n = b'$ with $b \in C_a$, $b' \notin C_a$. Then, concatenating with our sequence from a to b, we get a sequence from a to b', contradicting the assumption $b' \notin C_a$. Thus, the second condition holds, and C_a is a cell. So a lies in at least one cell.

   But now, note that if C is a cell containing a, then all b for which such a sequence from a to b exists must lie in C (or the second condition is violated), and if no such sequence exists, then b cannot lie in C (or the first condition is violated). Thus, the elements of C are uniquely determined, so there is exactly one cell containing a, and the proof is complete.

8. (a) Given a set A with $n \geq 1$ elements, find the number of consistent 2-configurations of A of order 1 with exactly 1 cell.

   Solution: There must be some pair $\{a, b\}$ in the 2-configuration, since each element $a \in A$ must belong to one pair. Since neither a nor b can now belong to any other pair, this must be the entire cell. Thus, there is 1 such 2-configuration when $n = 2$, and there are none when $n \neq 2$.

   (b) Given a set A with 10 elements, find the number of consistent 2-configurations of A of order 2 with exactly 1 cell.

   Solution: Consider such a configuration; let $\{a_1, a_2\}$ be an element of it. Then a_2 belongs to exactly one other pair; call it $\{a_2, a_3\}$. Likewise, a_3 belongs to exactly one other pair $\{a_3, a_4\}$, and so forth; since we have finitely many elements, we must eventually reach some pair $\{a_m, a_k\}$ that revisits a previously used element ($m > k$). But this is only possible if $k = 1$, since each other a_k with $k < m$ is already used in two pairs. Now, $\{a_1, \ldots, a_m\}$ constitutes a complete cell, because none of these elements can be used in any more pairs, so $m = 10$. Thus, every consistent 2-configuration of order 2 with exactly 1 cell gives rise to a permutation $a_1, \ldots, a_{10}$ of the elements of A, and conversely, each such permutation gives us a 2-configuration $\{\{a_1, a_2\}, \ldots, \{a_9, a_{10}\}, \{a_{10}, a_1\}\}$. In fact, each configuration corresponds to exactly 20 permutations, depending which of the

10 elements of the 2-configuration we choose as $\{a_1, a_2\}$ and which of these two elements of A we in turn choose to designate as a_1 (as opposed to a_2). Therefore, the number of such 2-configurations is $10!/20 = 9!/2 = 181440$. (More generally, the same argument shows that, when A has $n \geq 3$ elements, there are $(n-1)!/2$ such 2-configurations.)

(c) Given a set A with 10 elements, find the number of consistent 2-configurations of order 2 with exactly 2 cells.

Solution: Notice that if we look only at the pairs contained within any fixed cell, each element of that cell still lies in 2 such pairs, since all the pairs it belongs to are contained within that cell. Thus we have an induced consistent 2-configuration of order 2 of each cell.

Now, each cell must have at least 3 elements for the configuration to be 2-consistent. So we can have either two 5-element cells, a 4-element cell and a 6-element cell, or a 3-element cell and a 7-element cell. If there are two 5-element cells, we can choose the members of the first cell in $\binom{10}{5}$ ways, and then (by the reasoning in the previous problem) we have $4!/2$ ways to build a consistent 2-configuration of order 2 of each cell. However, choosing 5 elements for the first cell is equivalent to choosing the other 5 elements for the first cell, since the two cells are indistinguishable; thus, we have overcounted by a factor of 2. So we have $\binom{10}{5} \cdot (4!/2)^2/2 = 252 \cdot 144/2 = 18144$ ways to form our configuration if we require it to have two cells of 5 elements each.

If we have one 4-element cell and one 6-element cell, then there are $\binom{10}{4}$ ways to determine which 4 elements go in the smaller cell, and then $3!/2$ ways and $5!/2$ ways, respectively, to construct the 2-configurations of the two cells, for a total of $\binom{10}{4} \cdot (3!/2) \cdot (5!/2) = 210 \cdot 3 \cdot 60 = 37800$ configurations (no overcounting here), and by similar reasoning, we have $\binom{10}{3} \cdot (2!/2) \cdot (6!/2) = 120 \cdot 1 \cdot 360 = 43200$ configurations with one 3-element cell and one 7-element cell. Thus, altogether, we have a total of $18144 + 37800 + 43200 = 99144$ consistent 2-configurations of order 2 with exactly 2 cells.

9. (a) Show that if every cell of a 2-configuration of a finite set A is m-separable, then the whole 2-configuration is m-separable.

Solution: Let C be a 2-configuration of A with cells $A_1, \ldots, A_n$, so that there is no element of C with one element in A_i and another in A_j for $i \neq j$. Suppose that each cell is m-separable, so that for each i, $1 \leq i \leq n$, there is a labeling function $f_i : A_i \to \{1, \ldots, m\}$ such that no two elements in the same pair of C are assigned the same number. Then, by combining, we get a function f on all of A whose restriction to A_i is f_i for each i. By the definition of f_i, within each A_i there is no element of C both of whose elements are mapped to the same integer; and as above, we know that there are no elements of C not contained inside any A_i. Thus, C is m-separable, by the existence of f.

(b) Show that any barren 2-configuration of a finite set A is 2-separable.

Solution: It is sufficient to show each cell is 2-separable, by part (a). A barren 2-configuration by definition cannot have any cycles (i.e. subsets $\{a_0, \ldots, a_n\}$, $n \geq 2$, where each $\{a_i, a_{i+1}\}$ and $\{a_n, a_0\}$ all belong to the 2-configuration). For any two distinct elements a, b of A in the same cell of a 2-configuration C, define the *distance* between them to be the smallest n such that there exists a sequence $a = a_0, a_1, \ldots, a_n =$

b with $\{a_0, a_1\}, \{a_1, a_2\}, \dots, \{a_{n-1}, a_n\}$ all belonging to the 2-configuration. Notice that the terms of this sequence are all distinct: if $a_i = a_j$ for $i < j$, then we have the shorter sequence $a_0, a_1, \dots, a_i, a_{j+1}, \dots, a_n$, contradicting minimality.

Now let C be a barren 2-configuration of A. Pick any element a of A; label it and all elements at even distance from it with the integer 1, and label all elements at odd distance from it with the integer 2. We claim no two different elements with the same label appear in the same element of C. Otherwise, let b and c be such elements, and let $a = a_0, a_1, \dots, a_n = b$ and $a = a'_0, a'_1, \dots, a'_m = c$ be the corresponding minimal sequences. Consider the largest p such that $a_p \in \{a'_0, \dots, a'_m\}$; write $a_p = a'_q$. We claim the set $\{a_n, a_{n-1}, \dots, a_p, a'_{q+1}, \dots, a'_m\}$ is then a cycle. It is straightforward to check that all its elements are distinct; the only issue is whether it has at least 3 elements. If not, we would have $a_p = a_n$ or a'_m. Assume that $a_p = a_n \Rightarrow p = n \Rightarrow q = m - 1$ (by minimality of our sequence (a'_i)), but this means that $m = n + 1$, so the distances of b and c from a have opposite parities, contradicting the assumption that they have the same label. The case $a'_q = a'_m$ is similar. Thus, our set really does have at least three elements, and it is a cycle. But since A is barren, it contains no cycles, and we have a contradiction.

Thus, after all, no two elements with the same label appear in the same pair of C, so the cell containing a is 2-separable, and we are done.

10. Show that every consistent 2-configuration of order 4 on a finite set A has a subset that is a consistent 2-configuration of order 2.

Solution: First, assume the 2-configuration has just one cell. We claim there exists a sequence $a_0, a_1, \dots, a_n$ of elements of A (not necessarily all distinct) such that the list

$$\{a_0, a_1\}, \{a_1, a_2\}, \dots, \{a_{n-1}, a_n\}, \{a_n, a_0\}$$

contains each element of the 2-configuration exactly once. To see this, consider the longest sequence such that $\{a_0, a_1\}, \dots, \{a_{n-1}, a_n\}, \{a_n, a_0\}$ are all distinct elements of the 2-configuration. (We may take $n = 0$ if necessary. Note that the finiteness condition ensures such a maximal sequence exists.) Each element of A occurs an even number of times among these pairs (since each occurrence in the sequence contributes to two pairs). If every element occurs 4 times or 0 times, then the elements occurring in the sequence form a cell, since they cannot occur in any other pairs in the 2-configuration. Hence, they are all of A, and our sequence uses all the pairs in the 2-configuration, so the claim follows. Otherwise, there is some element a_i occurring exactly twice. Choose b_1 so that $\{a_i, b_1\}$ is one of the two pairs in the 2-configuration not used by our sequence. Then choose b_2 so that $\{b_1, b_2\}$ be another pair not used thus far. Continue in this manner, choosing new elements b_k with $\{b_k, b_{k+1}\}$ a pair not already used, until we reach a point where finding another unused pair is impossible. Now, our pairs so far are

$$\{a_0, a_1\}, \dots, \{a_{n-1}, a_n\}, \{a_n, a_0\},$$

$$\{a_i, b_1\}, \{b_1, b_2\}, \dots, \{b_{k-1}, b_k\}.$$

Every element is used in an even number of these pairs, except possibly a_i, which is used in three pairs, and b_k, which is used in an odd number of pairs (so one or three) — unless $a_i = b_k$, in which case this element occurs four times. But since it is impossible to continue the sequence, b_k must indeed have been used four times, so $b_k = a_i$.

But now we can construct the following sequence of distinct elements of the 2-configuration:

$$\{a_0, a_1\}, \ldots, \{a_{i-1}, a_i\}, \{a_i, b_1\}, \{b_1, b_2\}, \ldots, \{b_{k-1}, a_i\},$$

$$\{a_i, a_{i+1}\}, \ldots, \{a_{n-1}, a_n\}, \{a_n, a_0\}.$$

This contradicts the maximality of our original sequence. This contradiction means that our original sequence must have used all the pairs in the 2-configuration, after all.

So we can express the 2-configuration via such a sequence of pairs, where each pair's second element equals the first element of the next pair. If A has n elements, then (since each element appears in four pairs) we have $2n$ pairs. So we can choose the 1st, 3rd, 5th, $\ldots$, $(2n-1)$th pairs, and then each element of A belongs to just two of these pairs, because each occurrence of the element as an a_i contributes to two consecutive pairs from our original sequence (or the first and last such pairs). Thus, we have our consistent 2-configuration of order 2, as desired.

Finally, if A consists of more than one cell, then the pairs within any given cell form a consistent 2-configuration of order 4 on that cell. So we simply apply the above procedure to obtain a consistent 2-configuration of order 2 on each cell, and then combining these gives a consistent 2-configuration of order 2 on A, as desired.

Comments: A note for those who might have found these problems rather foreign — the objects described here are actually of considerable importance; they constitute the elements of graph theory, one of the major research areas of modern mathematics. What we have called a "2-configuration" is generally called a *graph*, and what we have called a "k-configuration" ($k > 2$) is generally called a *hypergraph*. The graph in problem 3b is the *Petersen graph*, a ubiquitous counterexample in graph theory. A consistent 2-configuration of order n is an *n-regular* graph; a cell is a *component*; a barren 2-configuration is a *forest* (and a forest with one component is a *tree*); and an m-separable configuration is *m-colorable* (and the minimum m for which a graph is m-colorable is its *chromatic number*).