# Harvard-MIT Mathematics Tournament

**February 28, 2004**

**Individual Round: Calculus Subject Test — Solutions**

1. Let $f(x) = \sin(\sin x)$. Evaluate $\lim_{h\to 0} \frac{f(x+h)-f(h)}{x}$ at $x = \pi$.

   **Solution:** $\boxed{0}$

   The expression $\frac{f(x+h)-f(h)}{x}$ is continuous at $h = 0$, so the limit is just $\frac{f(x)-f(0)}{x}$. Letting $x = \pi$ yields $\frac{\sin(\sin\pi)-\sin(\sin 0)}{\pi} = 0$.

2. Suppose the function $f(x) - f(2x)$ has derivative 5 at $x = 1$ and derivative 7 at $x = 2$. Find the derivative of $f(x) - f(4x)$ at $x = 1$.

   **Solution:** $\boxed{19}$

   Let $g(x) = f(x) - f(2x)$. Then we want the derivative of

   $$f(x) - f(4x) = (f(x) - f(2x)) + (f(2x) - f(4x)) = g(x) + g(2x)$$

   at $x = 1$. This is $g'(x) + 2g'(2x)$ at $x = 1$, or $5 + 2 \cdot 7 = 19$.

3. Find $\lim_{x\to\infty}(\sqrt[3]{x^3+x^2} - \sqrt[3]{x^3-x^2})$.

   **Solution:** $\boxed{2/3}$

   Observe that

   $$\lim_{x\to\infty}\left[(x+1/3) - \sqrt[3]{x^3+x^2}\right] = \lim_{x\to\infty}\frac{x/3+1/27}{(\sqrt[3]{x^3+x^2})^2 + (\sqrt[3]{x^3+x^2})(x+1/3) + (x+1/3)^2},$$

   by factoring the numerator as a difference of cubes. The numerator is linear in $x$, while the denominator is at least $3x^2$, so the limit as $x \to \infty$ is 0. By similar arguments, $\lim_{x\to\infty}[(x-1/3) - \sqrt[3]{x^3-x^2}] = 0$. So, the desired limit equals

   $$2/3 + \lim_{x\to\infty}[(x-1/3) - \sqrt[3]{x^3-x^2}] - \lim_{x\to\infty}[(x+1/3) - \sqrt[3]{x^3+x^2}] = 2/3.$$

4. Let $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos x)))))))$, and suppose that the number $a$ satisfies the equation $a = \cos a$. Express $f'(a)$ as a polynomial in $a$.

   **Solution:** $\boxed{a^8 - 4a^6 + 6a^4 - 4a^2 + 1}$

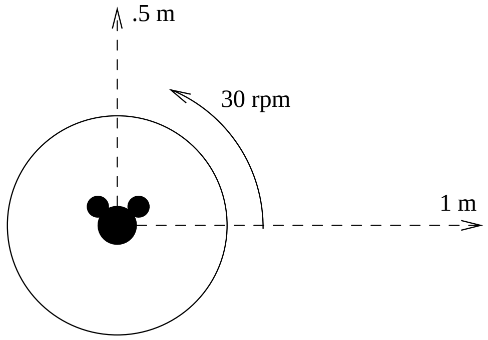
   This is an exercise using the chain rule. Define $f_0(x) = x$ and $f_n(x) = \cos f_{n-1}(x)$ for $n \geq 0$. We will show by induction that $f_n(a) = a$ and $f_n'(a) = (-\sin a)^n$ for all $n$. The case $n = 0$ is clear. Then $f_n(a) = \cos f_{n-1}(a) = \cos a = a$, and

   $$f_n'(a) = f_{n-1}'(a) \cdot (-\sin f_{n-1}(a)) = (-\sin a)^{n-1} \cdot (-\sin a) = (-\sin a)^n$$

   by induction. Now, $f(x) = f_8(x)$, so $f'(a) = (-\sin a)^8 = \sin^8 a$. But $\sin^2 a = 1 - \cos^2 a = 1 - a^2$, so $f'(a) = (1-a^2)^4 = a^8 - 4a^6 + 6a^4 - 4a^2 + 1$.

5. A mouse is sitting in a toy car on a negligibly small turntable. The car cannot turn on its own, but the mouse can control when the car is launched and when the car stops (the car has brakes). When the mouse chooses to launch, the car will immediately leave the turntable on a straight trajectory at 1 meter per second.

   Suddenly someone turns on the turntable; it spins at 30 rpm. Consider the set $S$ of points the mouse can reach in his car within 1 second after the turntable is set in motion. (For example, the arrows in the figure below represent two possible paths the mouse can take.) What is the area of $S$, in square meters?




   **Solution:** $\boxed{\pi/6}$

   The mouse can wait while the table rotates through some angle $\theta$ and then spend the remainder of the time moving along that ray at 1 m/s. He can reach any point between the starting point and the furthest reachable point along the ray, $(1-\theta/\pi)$ meters out. So the area is given by the polar integral

$$\int_0^\pi \frac{(1-\theta/\pi)^2}{2}\,d\theta = \frac{1}{2}\cdot\frac{1}{\pi^2}\int_0^\pi \phi^2\,d\phi = \pi/6$$

   (where we have used the change of variables $\phi = \pi - \theta$).

6. For $x > 0$, let $f(x) = x^x$. Find all values of $x$ for which $f(x) = f'(x)$.

   **Solution:** $\boxed{1}$

   Let $g(x) = \log f(x) = x\log x$. Then $f'(x)/f(x) = g'(x) = 1 + \log x$. Therefore $f(x) = f'(x)$ when $1 + \log x = 1$, that is, when $x = 1$.

7. Find the area of the region in the $xy$-plane satisfying $x^6 - x^2 + y^2 \le 0$.

   **Solution:** $\boxed{\pi/2}$

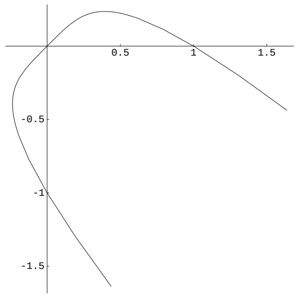
   Rewrite the condition as $|y| \le \sqrt{x^2 - x^6}$. The right side is zero when $x$ is $-1$, 0, or 1, and it bounds an area symmetric about the $x$- and $y$-axes. Therefore, we can calculate the area by the integral

$$2\int_{-1}^{1}\sqrt{x^2-x^6}\,dx = 4\int_0^1 x\sqrt{1-x^4}\,dx = 2\int_0^1\sqrt{1-u^2}\,du = \pi/2.$$

8. If $x$ and $y$ are real numbers with $(x+y)^4 = x - y$, what is the maximum possible value of $y$?

   **Solution:** $\boxed{3\sqrt[3]{2}/16}$

By drawing the graph of the curve (as shown), which is just a 135° clockwise rotation and scaling of $y = x^4$, we see that the maximum is achieved at the unique point where $dy/dx = 0$. Implicit differentiation gives $4(dy/dx + 1)(x + y)^3 = 1 - dy/dx$, so setting $dy/dx = 0$ gives $4(x+y)^3 = 1$. So $x+y = 1/\sqrt[3]{4} = \sqrt[3]{2}/2$, and $x-y = (x+y)^4 = \sqrt[3]{2}/8$. Subtracting and dividing by 2 gives $y = (\sqrt[3]{2}/2 - \sqrt[3]{2}/8)/2 = 3\sqrt[3]{2}/16$.




9. Find the positive constant $c_0$ such that the series

$$\sum_{n=0}^{\infty} \frac{n!}{(cn)^n}$$

converges for $c > c_0$ and diverges for $0 < c < c_0$.

**Solution:** $\boxed{1/e}$

The ratio test tells us that the series converges if

$$\lim_{n\to\infty} \frac{(n+1)!/(c(n+1))^{n+1}}{n!/(cn)^n} = \frac{1}{c} \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

is less than one and diverges if it is greater than one. But

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} = \frac{1}{e} .$$

Then the limit above is just $1/ce$, so the series converges for $c > 1/e$ and diverges for $0 < c < 1/e$.

10. Let $P(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$. Let $P^{[1]}(x) = P(x)$, and for $n \geq 1$, let $P^{[n+1]}(x) = P^{[n]}(P(x))$. Evaluate $\int_0^1 P^{[2004]}(x)\,dx$.

**Solution:** $\boxed{1/2}$

By Note that $P(1-x) = 1 - P(x)$. It follows easily by induction that $P^{[k]}(1-x) = 1 - P^{[k]}(x)$ for all positive integers $k$. Hence

$$\begin{aligned}\int_0^1 P^{[2004]}(x)\,dx &= \int_0^1 1 - P^{[2004]}(1-x)\,dx \\ &= 1 - \int_0^1 P^{[2004]}(1-x)\,dx \\ &= 1 - \int_0^1 P^{[2004]}(u)\,du \qquad (u = 1-x).\end{aligned}$$

Hence $\int_0^1 P^{[2004]}(x)\,dx = 1/2$.