Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Combinatorics Subject Test — Solutions

1. There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different rooms will have people in them?

   Solution: $\boxed{31}$

   We can prove by induction on n that the following pattern holds for $0 \leq n \leq 499$: after $2n$ minutes, the first room contains $1000 - 2n$ people and the next n rooms each contain 2 people, and after $2n + 1$ minutes, the first room contains $1000 - (2n + 1)$ people, the next n rooms each contain 2 people, and the next room after that contains 1 person. So, after 60 minutes, we have one room with 940 people and 30 rooms with 2 people each.

2. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

   Solution: $\boxed{21600}$

   In the top row, you can mark any of the 6 squares that is not a corner. In the bottom row, you can then mark any of the 5 squares that is not a corner and not in the same column as the square just marked. Then, in the second row, you have 6 choices for a square not in the same column as either of the two squares already marked; then there are 5 choices remaining for the third row, and so on down to 1 for the seventh row, in which you make the last mark. Thus, altogether, there are $6 \cdot 5 \cdot (6 \cdot 5 \cdots 1) = 30 \cdot 6! = 30 \cdot 720 = 21600$ possible sets of squares.

3. A class of 10 students took a math test. Each problem was solved by exactly 7 of the students. If the first nine students each solved 4 problems, how many problems did the tenth student solve?

   Solution: $\boxed{6}$

   Suppose the last student solved n problems, and the total number of problems on the test was p. Then the total number of correct solutions written was $7p$ (seven per problem), and also equal to $36 + n$ (the sum of the students' scores), so $p = (36 + n)/7$. The smallest $n \geq 0$ for which this is an integer is $n = 6$. But we also must have $n \leq p$, so $7n \leq 36 + n$, and solving gives $n \leq 6$. Thus $n = 6$ is the answer.

4. Andrea flips a fair coin repeatedly, continuing until she either flips two heads in a row (the sequence HH) or flips tails followed by heads (the sequence TH). What is the probability that she will stop after flipping HH?

   Solution: $\boxed{1/4}$

The only way that Andrea can ever flip HH is if she never flips T, in which case she must flip two heads immediately at the beginning. This happens with probability $\frac{1}{4}$.

5. A best-of-9 series is to be played between two teams; that is, the first team to win 5 games is the winner. The Mathletes have a chance of 2/3 of winning any given game. What is the probability that exactly 7 games will need to be played to determine a winner?

   Solution: $\boxed{20/81}$

   If the Mathletes are to win, they must win exactly 5 out of the 7 games. One of the 5 games they win must be the 7th game, because otherwise they would win the tournament before 7 games are completed. Thus, in the first 6 games, the Mathletes must win 4 games and lose 2. The probability of this happening and the Mathletes winning the last game is

   $$\left[\binom{6}{2} \cdot \left(\frac{2}{3}\right)^4 \cdot \left(\frac{1}{3}\right)^2\right] \cdot \left(\frac{2}{3}\right).$$

   Likewise, the probability of the other team winning on the 7th game is

   $$\left[\binom{6}{2} \cdot \left(\frac{1}{3}\right)^4 \cdot \left(\frac{2}{3}\right)^2\right] \cdot \left(\frac{1}{3}\right).$$

   Summing these values, we obtain $160/729 + 20/729 = 20/81$.

6. A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Bill and Karl must serve together or not at all, and Alice and Jane refuse to serve with each other?

   Solution: $\boxed{41}$

   If Bill and Karl are on the committee, there are $\binom{7}{3} = 35$ ways for the other group members to be chosen. However, if Alice and Jane are on the committee with Bill and Karl, there are $\binom{5}{1} = 5$ ways for the last member to be chosen, yielding 5 unacceptable committees. If Bill and Karl are not on the committee, there are $\binom{7}{5} = 21$ ways for the 5 members to be chosen, but again if Alice and Jane were to be on the committee, there would be $\binom{5}{3} = 10$ ways to choose the other three members, yielding 10 more unacceptable committees. So, we obtain $(35 - 5) + (21 - 10) = 41$ ways the committee can be chosen.

7. We have a polyhedron such that an ant can walk from one vertex to another, traveling only along edges, and traversing every edge exactly once. What is the smallest possible total number of vertices, edges, and faces of this polyhedron?

   Solution: $\boxed{20}$

   This is obtainable by construction. Consider two tetrahedrons glued along a face; this gives us 5 vertices, 9 edges, and 6 faces, for a total of 20, and one readily checks that the required Eulerian path exists. Now, to see that we cannot do better, first notice that the number v of vertices is at least 5, since otherwise we must have a tetrahedron, which does not have an Eulerian path. Each vertex is incident to at least 3 edges, and in fact, since there is an Eulerian path, all except possibly two vertices are incident to an even number of edges. So the number of edges is at least $(3+3+4+4+4)/2$ (since each edge meets two vertices) $= 9$. Finally, if $f = 4$ then each face must be a triangle, because there are only 3 other faces for it to share edges with, and we are again in the case of a tetrahedron, which is impossible; therefore $f \geq 5$. So $f+v+e \geq 5+5+9 = 19$. But since $f+v-e = 2-2g$ (where g is the number of holes in the polyhedron), $f+v+e$ must be even. This strengthens our bound to 20 as needed.

8. Urn A contains 4 white balls and 2 red balls. Urn B contains 3 red balls and 3 black balls. An urn is randomly selected, and then a ball inside of that urn is removed. We then repeat the process of selecting an urn and drawing out a ball, without returning the first ball. What is the probability that the first ball drawn was red, given that the second ball drawn was black?

   Solution: $\boxed{7/15}$

   This is a case of conditional probability; the answer is the probability that the first ball is red and the second ball is black, divided by the probability that the second ball is black.

   First, we compute the numerator. If the first ball is drawn from Urn A, we have a probability of $2/6$ of getting a red ball, then a probability of $1/2$ of drawing the second ball from Urn B, and a further probability of $3/6$ of drawing a black ball. If the first ball is drawn from Urn B, we have probability $3/6$ of getting a red ball, then $1/2$ of drawing the second ball from Urn B, and $3/5$ of getting a black ball. So our numerator is

$$\frac{1}{2}\left(\frac{2}{6}\cdot\frac{1}{2}\cdot\frac{3}{6}+\frac{3}{6}\cdot\frac{1}{2}\cdot\frac{3}{5}\right)=\frac{7}{60}.$$

   We similarly compute the denominator: if the first ball is drawn from Urn A, we have a probability of $1/2$ of drawing the second ball from Urn B, and $3/6$ of drawing a black ball. If the first ball is drawn from Urn B, then we have probability $3/6$ that it is red, in which case the second ball will be black with probability $(1/2)\cdot(3/5)$, and probability $3/6$ that the first ball is black, in which case the second is black with probability $(1/2)\cdot(2/5)$. So overall, our denominator is

$$\frac{1}{2}\left(\frac{1}{2}\cdot\frac{3}{6}+\frac{3}{6}\left[\frac{1}{2}\cdot\frac{3}{5}+\frac{1}{2}\cdot\frac{2}{5}\right]\right)=\frac{1}{4}.$$

   Thus, the desired conditional probability is $(7/60)\,/\,(1/4) = 7/15$.

9. A classroom consists of a 5×5 array of desks, to be filled by anywhere from 0 to 25 students, inclusive. No student will sit at a desk unless either all other desks in its row or all others in its column are filled (or both). Considering only the set of desks that are occupied (and not which student sits at each desk), how many possible arrangements are there?

   Solution: $\boxed{962}$

   The set of empty desks must be of the form (non-full rows) $\times$ (non-full columns): each empty desk is in a non-full column and a non-full row, and the given condition implies that each desk in such a position is empty. So if there are fewer than 25 students, then both of these sets are nonempty; we have $2^5 - 1 = 31$ possible sets of non-full rows, and 31 sets of non-full columns, for 961 possible arrangements. Alternatively, there may be 25 students, and then only 1 arrangement is possible. Thus there are 962 possibilities altogether.

10. In a game similar to three card monte, the dealer places three cards on the table: the queen of spades and two red cards. The cards are placed in a row, and the queen starts in the center; the card configuration is thus RQR. The dealer proceeds to move. With each move, the dealer randomly switches the center card with one of the two edge cards (so the configuration after the first move is either RRQ or QRR). What is the probability that, after 2004 moves, the center card is the queen?

    Solution: $\boxed{1/3 + 1/(3 \cdot 2^{2003})}$

    If the probability that the queen is the center card after move n is p_n, then the probability that the queen is an edge card is $1 - p_n$, and the probability that the queen is the center card after move $n + 1$ is $p_{n+1} = (1 - p_n)/2$. This recursion allows us to calculate the first few values of p_n. We might then notice in $1, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \frac{11}{32}, \cdots$, that the value of each fraction is close to 1/3, and getting closer for larger n. In fact subtracting 1/3 from each fraction yields $\frac{2}{3}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{12}, \frac{1}{24}, -\frac{1}{48}, \cdots$. This suggests the formula $p_n = \frac{1}{3} + \frac{2}{3}(-\frac{1}{2})^n$, and one can then prove that this formula is in fact correct by induction. Thus, $p(2004) = \frac{1}{3} + \frac{2}{3}(-\frac{1}{2})^{2004} = \frac{1}{3} + \frac{1}{3 \cdot 2^{2003}}$.

    The recurrence can also be solved without guessing — by generating functions, for example, or by using the fundamental theorem of linear recurrences, which ensures that the solution is of the form $p_n = a + b(-1/2)^n$ for some constants a, b.