Harvard-MIT Mathematics Tournament

February 28, 2004

Guts Round — Solutions

1. Find the value of

$$\binom{6}{1}2^1+\binom{6}{2}2^2+\binom{6}{3}2^3+\binom{6}{4}2^4+\binom{6}{5}2^5+\binom{6}{6}2^6.$$

Solution: $\boxed{728}$

This sum is the binomial expansion of $(1+2)^6$, except that it is missing the first term, $\binom{6}{0}2^0=1$. So we get $3^6-1=728$.

2. If the three points

$$(1,a,b)$$
$$(a,2,b)$$
$$(a,b,3)$$

are collinear (in 3-space), what is the value of $a+b$?

Solution: $\boxed{4}$

The first two points are distinct (otherwise we would have $a=1$ and $a=2$ simultaneously), and they both lie on the plane $z=b$, so the whole line is in this plane and $b=3$. Reasoning similarly with the last two points gives $a=1$, so $a+b=4$.

3. If the system of equations

$$\begin{aligned}|x+y| &= 99\\ |x-y| &= c\end{aligned}$$

has exactly two real solutions (x,y), find the value of c.

Solution: $\boxed{0}$

If $c<0$, there are no solutions. If $c>0$ then we have four possible systems of linear equations given by $x+y=\pm 99$, $x-y=\pm c$, giving four solutions (x,y). So we must have $c=0$, and then we do get two solutions ($x=y$, so they must both equal $\pm 99/2$).

4. A tree grows in a rather peculiar manner. Lateral cross-sections of the trunk, leaves, branches, twigs, and so forth are circles. The trunk is 1 meter in diameter to a height of 1 meter, at which point it splits into two sections, each with diameter .5 meter. These sections are each one meter long, at which point they each split into two sections, each with diameter .25 meter. This continues indefinitely: every section of tree is 1 meter long and splits into two smaller sections, each with half the diameter of the previous.

What is the total volume of the tree?

Solution: $\boxed{\pi/2}$

If we count the trunk as level 0, the two sections emerging from it as level 1, and so forth, then the nth level consists of 2^n sections each with diameter $1/2^n$, for a volume of $2^n(\pi/4\cdot 2^{-2n})=(\pi/4)\cdot 2^{-n}$. So the total volume is given by a simple infinite sum,

$$.25\pi\cdot(1+1/2+1/4+...)=.25\pi\cdot 2=\pi/2.$$

5. Augustin has six $1 \times 2 \times \pi$ bricks. He stacks them, one on top of another, to form a tower six bricks high. Each brick can be in any orientation so long as it rests flat on top of the next brick below it (or on the floor). How many distinct heights of towers can he make?

Solution: $\boxed{28}$

If there are k bricks which are placed so that they contribute either 1 or 2 height, then the height of these k bricks can be any integer from k to $2k$. Furthermore, towers with different values of k cannot have the same height. Thus, for each k there are $k+1$ possible tower heights, and since k is any integer from 0 to 6, there are $1+2+3+4+5+6+7=28$ possible heights.

6. Find the smallest integer n such that $\sqrt{n+99}-\sqrt{n}<1$.

Solution: $\boxed{2402}$

This is equivalent to

$$\begin{aligned} \sqrt{n+99} &< \sqrt{n}+1 \\ n+99 &< n+1+2\sqrt{n} \\ 49 &< \sqrt{n} \end{aligned}$$

So the smallest integer n with this property is $49^2+1=2402$.

7. Find the shortest distance from the line $3x+4y=25$ to the circle $x^2+y^2=6x-8y$.

Solution: $\boxed{7/5}$

The circle is $(x-3)^2+(y+4)^2=5^2$. The center $(3,-4)$ is a distance of

$$\frac{|3\cdot 3+4\cdot -4-25|}{\sqrt{3^2+4^2}}=\frac{32}{5}$$

from the line, so we subtract 5 for the radius of the circle and get $7/5$.

8. I have chosen five of the numbers $\{1,2,3,4,5,6,7\}$. If I told you what their product was, that would not be enough information for you to figure out whether their sum was even or odd. What is their product?

Solution: $\boxed{420}$

Giving you the product of the five numbers is equivalent to telling you the product of the two numbers I didn't choose. The only possible products that are achieved by more than one pair of numbers are 12 ($\{3,4\}$ and $\{2,6\}$) and 6 ($\{1,6\}$ and $\{2,3\}$). But in the second case, you at least know that the two unchosen numbers have odd sum (and so the five chosen numbers have odd sum also). Therefore, the first case must hold, and the product of the five chosen numbers is

$$1\cdot 2\cdot 5\cdot 6\cdot 7=1\cdot 3\cdot 4\cdot 5\cdot 7=420.$$

9. A positive integer n is *picante* if $n!$ ends in the same number of zeroes whether written in base 7 or in base 8. How many of the numbers $1,2,\ldots,2004$ are picante?

Solution: $\boxed{4}$

The number of zeroes in base 7 is the total number of factors of 7 in $1 \cdot 2 \cdots n$, which is

$$\lfloor n/7 \rfloor + \lfloor n/7^2 \rfloor + \lfloor n/7^3 \rfloor + \cdots .$$

The number of zeroes in base 8 is $\lfloor a \rfloor$, where

$$a = (\lfloor n/2 \rfloor + \lfloor n/2^2 \rfloor + \lfloor n/2^3 \rfloor + \cdots)/3$$

is one-third the number of factors of 2 in the product $n!$. Now $\lfloor n/2^k \rfloor/3 \geq \lfloor n/7^k \rfloor$ for all k, since $(n/2^k)/3 \geq n/7^k$. But n can only be picante if the two sums differ by at most $2/3$, so in particular this requires $(\lfloor n/2^2 \rfloor)/3 \leq \lfloor n/7^2 \rfloor + 2/3 \Leftrightarrow \lfloor n/4 \rfloor \leq 3 \lfloor n/49 \rfloor + 2$. This cannot happen for $n \geq 12$; checking the remaining few cases by hand, we find $n = 1, 2, 3, 7$ are picante, for a total of 4 values.

10. Let $f(x) = x^2 + x^4 + x^6 + x^8 + \cdots$, for all real x such that the sum converges. For how many real numbers x does $f(x) = x$?

    Solution: $\boxed{2}$

    Clearly $x = 0$ works. Otherwise, we want $x = x^2/(1 - x^2)$, or $x^2 + x - 1 = 0$. Discard the negative root (since the sum doesn't converge there), but $(-1 + \sqrt{5})/2$ works, for a total of 2 values.

11. Find all numbers n with the following property: there is exactly one set of 8 different positive integers whose sum is n.

    Solution: $\boxed{36, 37}$

    The sum of 8 different positive integers is at least $1 + 2 + 3 + \cdots + 8 = 36$, so we must have $n \geq 36$. Now $n = 36$ satisfies the desired property, since in this case we must have equality — the eight numbers must be $1, \ldots, 8$. And if $n = 37$ the eight numbers must be $1, 2, \ldots, 7, 9$: if the highest number is 8 then the sum is $36 < n$, while if the highest number is more than 9 the sum is $> 1 + 2 + \cdots + 7 + 9 = 37 = n$. So the highest number must be 9, and then the remaining numbers must be $1, 2, \ldots, 7$. Thus $n = 37$ also has the desired property.

    However, no other values of n work: if $n > 37$ then $\{1, 2, 3, \ldots, 7, n - 28\}$ and $\{1, 2, \ldots, 6, 8, n - 29\}$ are both sets of 8 distinct positive integers whose sum is n. So $n = 36, 37$ are the only solutions.

12. A convex quadrilateral is drawn in the coordinate plane such that each of its vertices (x, y) satisfies the equations $x^2 + y^2 = 73$ and $xy = 24$. What is the area of this quadrilateral?

    Solution: $\boxed{110}$

    The vertices all satisfy $(x + y)^2 = x^2 + y^2 + 2xy = 73 + 2 \cdot 24 = 121$, so $x + y = \pm 11$. Similarly, $(x - y)^2 = x^2 + y^2 - 2xy = 73 - 2 \cdot 24 = 25$, so $x - y = \pm 5$. Thus, there are four solutions: $(x, y) = (8, 3), (3, 8), (-3, -8), (-8, -3)$. All four of these solutions satisfy the original equations. The quadrilateral is therefore a rectangle with side lengths of $5\sqrt{2}$ and $11\sqrt{2}$, so its area is 110.

13. Find all positive integer solutions (m, n) to the following equation:

$$m^2 = 1! + 2! + \cdots + n!.$$

Solution: $\boxed{(1,1),(3,3)}$

A square must end in the digit $0, 1, 4, 5, 6$, or 9. If $n \geq 4$, then $1! + 2! + \cdots + n!$ ends in the digit 3, so cannot be a square. A simple check for the remaining cases reveals that the only solutions are $(1, 1)$ and $(3, 3)$.

14. If $a_1 = 1, a_2 = 0$, and $a_{n+1} = a_n + \frac{a_{n+2}}{2}$ for all $n \geq 1$, compute a_{2004}.

Solution: $\boxed{-2^{1002}}$

By writing out the first few terms, we find that $a_{n+4} = -4a_n$. Indeed,

$$a_{n+4} = 2(a_{n+3} - a_{n+2}) = 2(a_{n+2} - 2a_{n+1}) = 2(-2a_n) = -4a_n.$$

Then, by induction, we get $a_{4k} = (-4)^k$ for all positive integers k, and setting $k = 501$ gives the answer.

15. A regular decagon $A_0A_1A_2 \cdots A_9$ is given in the plane. Compute $\angle A_0A_3A_7$ in degrees.

Solution: $\boxed{54^\circ}$

Put the decagon in a circle. Each side subtends an arc of $360^\circ/10 = 36^\circ$. The inscribed angle $\angle A_0A_3A_7$ contains 3 segments, namely A_7A_8, A_8A_9, A_9A_0, so the angle is $108^\circ/2 = 54^\circ$.

16. An n-string is a string of digits formed by writing the numbers $1, 2, \ldots, n$ in some order (in base ten). For example, one possible 10-string is

$$35728910461$$

What is the smallest $n > 1$ such that there exists a palindromic n-string?

Solution: $\boxed{19}$

The following is such a string for $n = 19$:

$$9|18|7|16|5|14|3|12|1|10|11|2|13|4|15|6|17|8|19$$

where the vertical bars indicate breaks between the numbers. On the other hand, to see that $n = 19$ is the minimum, notice that only one digit can occur an odd number of times in a palindromic n-string (namely the center digit). If $n \leq 9$, then (say) the digits $1, 2$ each appear once in any n-string, so we cannot have a palindrome. If $10 \leq n \leq 18$, then $0, 9$ each appear once, and we again cannot have a palindrome. So 19 is the smallest possible n.

17. Kate has four red socks and four blue socks. If she randomly divides these eight socks into four pairs, what is the probability that none of the pairs will be mismatched? That is, what is the probability that each pair will consist either of two red socks or of two blue socks?

Solution: $\boxed{3/35}$

The number of ways Kate can divide the four red socks into two pairs is $\binom{4}{2}/2 = 3$. The number of ways she can divide the four blue socks into two pairs is also 3. Therefore, the number of ways she can form two pairs of red socks and two pairs of blue socks is

$3 \cdot 3 = 9$. The total number of ways she can divide the eight socks into four pairs is $[8!/(2! \cdot 2! \cdot 2! \cdot 2!)]/4! = 105$, so the probability that the socks come out paired correctly is $9/105 = 3/35$.

To see why 105 is the correct denominator, we can look at each 2! term as representing the double counting of pair (ab) and pair (ba), while the 4! term represents the number of different orders in which we can select the same four pairs. Alternatively, we know that there are three ways to select two pairs from four socks. To select three pairs from six socks, there are five different choices for the first sock's partner and then three ways to pair up the remaining four socks, for a total of $5 \cdot 3 = 15$ pairings. To select four pairs from eight socks, there are seven different choices for the first sock's partner and then fifteen ways to pair up the remaining six socks, for a total of $7 \cdot 15 = 105$ pairings.

18. On a spherical planet with diameter $10,000$ km, powerful explosives are placed at the north and south poles. The explosives are designed to vaporize all matter within $5,000$ km of ground zero and leave anything beyond $5,000$ km untouched. After the explosives are set off, what is the new surface area of the planet, in square kilometers?

    Solution: $\boxed{100,000,000\pi}$

    The explosives have the same radius as the planet, so the surface area of the "cap" removed is the same as the new surface area revealed in the resulting "dimple." Thus the area is preserved by the explosion and remains $\pi \cdot (10,000)^2$.

19. The Fibonacci numbers are defined by $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. If the number

    $$\frac{F_{2003}}{F_{2002}} - \frac{F_{2004}}{F_{2003}}$$

    is written as a fraction in lowest terms, what is the numerator?

    Solution: $\boxed{1}$

    Before reducing, the numerator is $F_{2003}^2 - F_{2002}F_{2004}$. We claim $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$, which will immediately imply that the answer is 1 (no reducing required). This claim is straightforward to prove by induction on n: it holds for $n = 2$, and if it holds for some n, then

    $$F_{n+1}^2 - F_nF_{n+2} = F_{n+1}(F_{n-1}+F_n) - F_n(F_n+F_{n+1}) = F_{n+1}F_{n-1} - F_n^2 = -(-1)^{n+1} = (-1)^{n+2}.$$

20. Two positive rational numbers x and y, when written in lowest terms, have the property that the sum of their numerators is 9 and the sum of their denominators is 10. What is the largest possible value of $x + y$?

    Solution: $\boxed{73/9}$

    For fixed denominators $a < b$ (with sum 10), we maximize the sum of the fractions by giving the smaller denominator as large a numerator as possible: $8/a + 1/b$. Then, if $a \geq 2$, this quantity is at most $8/2 + 1/1 = 5$, which is clearly smaller than the sum we get by setting $a = 1$, namely $8/1 + 1/9 = 73/9$. So this is the answer.

21. Find all ordered pairs of integers (x, y) such that $3^x 4^y = 2^{x+y} + 2^{2(x+y)-1}$.

    Solution: $\boxed{(0,1), (1,1), (2,2)}$

The right side is $2^{x+y}(1+2^{x+y-1})$. If the second factor is odd, it needs to be a power of 3, so the only options are $x+y=2$ and $x+y=4$. This leads to two solutions, namely (1,1) and (2,2). The second factor can also be even, if $x+y-1=0$. Then $x+y=1$ and $3^x4^y=2+2$, giving $(0,1)$ as the only other solution.

22. I have written a strictly increasing sequence of six positive integers, such that each number (besides the first) is a multiple of the one before it, and the sum of all six numbers is 79. What is the largest number in my sequence?

    Solution: $\boxed{48}$

    If the fourth number is ≥ 12, then the last three numbers must sum to at least $12+2\cdot 12+2^2\cdot 12=84>79$. This is impossible, so the fourth number must be less than 12. Then the only way we can have the required divisibilities among the first four numbers is if they are $1,2,4,8$. So the last two numbers now sum to $79-15=64$. If we call these numbers $8a, 8ab$ $(a,b>1)$ then we get $a(1+b)=a+ab=8$, which forces $a=2, b=3$. So the last two numbers are $16, 48$.

23. Find the largest integer n such that $3^{512}-1$ is divisible by 2^n.

    Solution: $\boxed{11}$

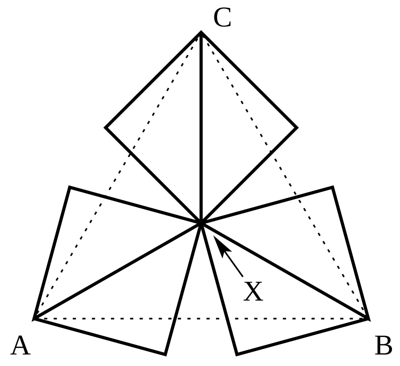
    Write
    $$\begin{aligned}3^{512}-1 &= (3^{256}+1)(3^{256}-1) = (3^{256}+1)(3^{128}+1)(3^{128}-1)\\ &= \cdots = (3^{256}+1)(3^{128}+1)\cdots(3+1)(3-1).\end{aligned}$$
    Now each factor $3^{2^k}+1$, $k\geq 1$, is divisible by just one factor of 2, since $3^{2^k}+1=(3^2)^{2^{k-1}}+1\equiv 1^{2^{k-1}}+1=2 \pmod 4$. Thus we get 8 factors of 2 here, and the remaining terms $(3+1)(3-1)=8$ give us 3 more factors of 2, for a total of 11.

24. We say a point is *contained* in a square if it is in its interior or on its boundary. Three unit squares are given in the plane such that there is a point contained in all three. Furthermore, three points A, B, C, are given, each contained in at least one of the squares. Find the maximum area of triangle ABC.

    Solution: $\boxed{3\sqrt{3}/2}$

    Let X be a point contained in all three squares. The distance from X to any point in any of the three squares is at most $\sqrt{2}$, the length of the diagonal of the squares. Therefore, triangle ABC is contained in a circle of radius $\sqrt{2}$, so its circumradius is at most $\sqrt{2}$. The triangle with greatest area that satisfies this property is the equilateral triangle in a circle of radius $\sqrt{2}$. (This can be proved, for example, by considering that the maximum altitude to any given side is obtained by putting the opposite vertex at the midpoint of its arc, and it follows that all the vertices are equidistant.) The equilateral triangle is also attainable, since making X the circumcenter and positioning the squares such that AX, BX, and CX are diagonals (of the three squares) and ABC is equilateral, leads to such a triangle. This triangle has area $3\sqrt{3}/2$, which may be calculated, for example, using the sine formula for area applied to ABX, ACX, and BCX, to get $3/2(\sqrt{2})^2\sin 120°$. (See diagram, next page.)

25. Suppose $x^3-ax^2+bx-48$ is a polynomial with three positive roots p, q, and r such that $p<q<r$. What is the minimum possible value of $1/p+2/q+3/r$?



Solution: $\boxed{3/2}$

We know $pqr = 48$ since the product of the roots of a cubic is the constant term. Now,

$$\frac{1}{p} + \frac{2}{q} + \frac{3}{r} \geq 3\sqrt[3]{\frac{6}{pqr}} = \frac{3}{2}$$

by AM-GM, with equality when $1/p = 2/q = 3/r$. This occurs when $p = 2$, $q = 4$, $r = 6$, so $3/2$ is in fact the minimum possible value.

26. How many of the integers $1, 2, \ldots, 2004$ can be represented as $(mn + 1)/(m + n)$ for positive integers m and n?

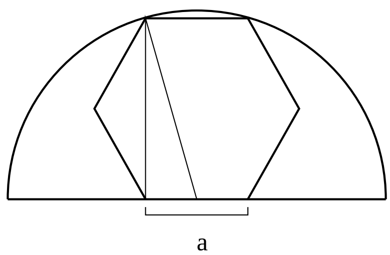
    Solution: $\boxed{2004}$

    For any positive integer a, we can let $m = a^2 + a - 1, n = a + 1$ to see that every positive integer has this property, so the answer is 2004.

27. A regular hexagon has one side along the diameter of a semicircle, and the two opposite vertices on the semicircle. Find the area of the hexagon if the diameter of the semicircle is 1.

    Solution: $\boxed{3\sqrt{3}/26}$

    The midpoint of the side of the hexagon on the diameter is the center of the circle. Draw the segment from this center to a vertex of the hexagon on the circle. This segment, whose length is $1/2$, is the hypotenuse of a right triangle whose legs have lengths $a/2$ and $a\sqrt{3}$, where a is a side of the hexagon. So $1/4 = a^2(1/4 + 3)$, so $a^2 = 1/13$. The hexagon consists of 6 equilateral triangles of side length a, so the area of the hexagon is $3a^2\sqrt{3}/2 = 3\sqrt{3}/26$.



28. Find the value of

$$\binom{2003}{1} + \binom{2003}{4} + \binom{2003}{7} + \cdots + \binom{2003}{2002}.$$

Solution: $\boxed{(2^{2003} - 2)/3}$

Let $\omega = -1/2 + i\sqrt{3}/2$ be a complex cube root of unity. Then, by the binomial theorem, we have

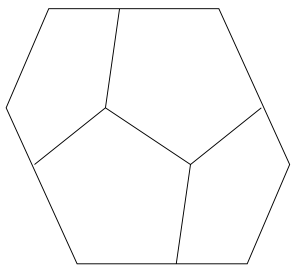
$$\begin{aligned}
\omega^2(\omega+1)^{2003} &= \binom{2003}{0}\omega^2 + \binom{2003}{1}\omega^3 + \binom{2003}{2}\omega^4 + \cdots + \binom{2003}{2003}\omega^{2005} \\
2^{2003} &= \binom{2003}{0} + \binom{2003}{1} + \binom{2003}{2} + \cdots + \binom{2003}{2003} \\
\omega^{-2}(\omega^{-1}+1)^{2003} &= \binom{2003}{0}\omega^{-2} + \binom{2003}{1}\omega^{-3} + \binom{2003}{2}\omega^{-4} + \cdots + \binom{2003}{2003}\omega^{-2005}
\end{aligned}$$

If we add these together, then the terms $\binom{2003}{n}$ for $n \equiv 1 \pmod 3$ appear with coefficient 3, while the remaining terms appear with coefficient $1 + \omega + \omega^2 = 0$. Thus the desired sum is just $(\omega^2(\omega+1)^{2003} + 2^{2003} + \omega^{-2}(\omega^{-1}+1)^{2003})/3$. Simplifying using $\omega + 1 = -\omega^2$ and $\omega^{-1} + 1 = -\omega$ gives $(-1 + 2^{2003} + -1)/3 = (2^{2003} - 2)/3$.

29. A regular dodecahedron is projected orthogonally onto a plane, and its image is an n-sided polygon. What is the smallest possible value of n?

Solution: $\boxed{6}$

We can achieve 6 by projecting onto a plane perpendicular to an edge of the dodecaheron. Indeed, if we imagine viewing the dodecahedron in such a direction, then 4 of the faces are projected to line segments (namely, the two faces adjacent to the edge and the two opposite faces), and of the remaining 8 faces, 4 appear on the front of the dodecahedron and the other 4 are on the back. Thus, the dodecahedron appears as shown.



To see that we cannot do better, note that, by central symmetry, the number of edges of the projection must be even. So we just need to show that the answer cannot be 4. But if the projection had 4 sides, one of the vertices would give a projection forming an acute angle, which is not possible. So 6 is the answer.

30. We have an n-gon, and each of its vertices is labeled with a number from the set $\{1, \ldots, 10\}$. We know that for any pair of distinct numbers from this set there is

at least one side of the polygon whose endpoints have these two numbers. Find the smallest possible value of n.

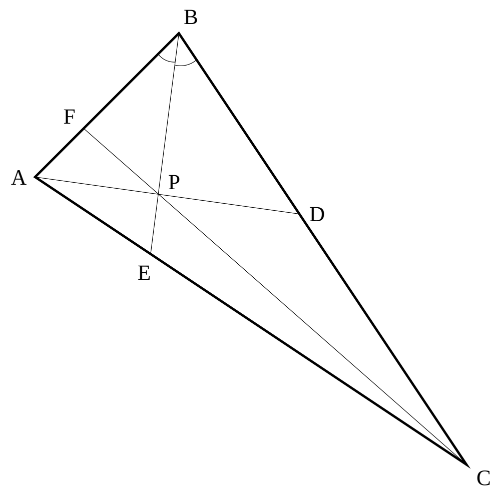
Solution: $\boxed{50}$

Each number be paired with each of the 9 other numbers, but each vertex can be used in at most 2 different pairs, so each number must occur on at least $\lceil 9/2 \rceil = 5$ different vertices. Thus, we need at least $10 \cdot 5 = 50$ vertices, so $n \geq 50$.

To see that $n = 50$ is feasible, let the numbers $1, \ldots, 10$ be the vertices of a complete graph. Then each vertex has degree 9, and there are $\binom{10}{2} = 45$ edges. If we attach extra copies of the edges 1-2, 3-4, 5-6, 7-8, and 9-10, then every vertex will have degree 10. In particular, the graph has an Eulerian tour, so we can follow this tour, successively numbering vertices of the 50-gon according to the vertices of the graph we visit. Then, for each edge of the graph, there will be a corresponding edge of the polygon with the same two vertex labels on its endpoints. It follows that every pair of distinct numbers occurs at the endpoints of some edge of the polygon, and so $n = 50$ is the answer.

31. P is a point inside triangle ABC, and lines AP, BP, CP intersect the opposite sides BC, CA, AB in points D, E, F, respectively. It is given that $\angle APB = 90°$, and that $AC = BC$ and $AB = BD$. We also know that $BF = 1$, and that $BC = 999$. Find AF.

Solution: $\boxed{499/500}$



Let $AC = BC = s$, $AB = BD = t$. Since BP is the altitude in isosceles triangle ABD, it bisects angle B. So, the Angle Bisector Theorem in triangle ABC given $AE/EC = AB/BC = t/s$. Meanwhile, $CD/DB = (s-t)/t$. Now Ceva's theorem gives us

$$\frac{AF}{FB} = \left(\frac{AE}{EC}\right) \cdot \left(\frac{CD}{DB}\right) = \frac{s-t}{s}$$

$$\Rightarrow \frac{AB}{FB} = 1 + \frac{s-t}{s} = \frac{2s-t}{s} \quad \Rightarrow \quad FB = \frac{st}{2s-t}.$$

Now we know $s = 999$, but we need to find t given that $st/(2s-t) = FB = 1$. So $st = 2s - t \Rightarrow t = 2s/(s+1)$, and then

$$AF = FB \cdot \frac{AF}{FB} = 1 \cdot \frac{s-t}{s} = \frac{(s^2 - s)/(s+1)}{s} = \frac{s-1}{s+1} = \frac{499}{500}.$$

32. Define the sequence $b_0, b_1, \ldots, b_{59}$ by

$$b_i = \begin{cases} 1 & \text{if i is a multiple of 3} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{a_i\}$ be a sequence of elements of $\{0, 1\}$ such that

$$b_n \equiv a_{n-1} + a_n + a_{n+1} \pmod 2$$

for $0 \le n \le 59$ ($a_0 = a_{60}$ and $a_{-1} = a_{59}$). Find all possible values of $4a_0 + 2a_1 + a_2$.

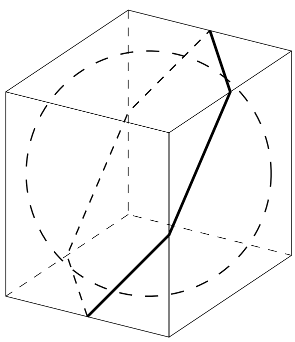
Solution: $\boxed{0, 3, 5, 6}$

Try the four possible combinations of values for a_0 and a_1. Since we can write $a_n \equiv b_{n-1} - a_{n-2} - a_{n-1}$, these two numbers completely determine the solution $\{a_i\}$ beginning with them (if there is one). For $a_0 = a_1 = 0$, we can check that the sequence beginning $0, 0, 0, 0, 1, 1$ and repeating every 6 indices is a possible solution for $\{a_i\}$, so one possible value for $4a_0 + 2a_1 + a_2$ is 0. The other three combinations for a_0 and a_1 similarly lead to valid sequences (produced by repeating the sextuples $0, 1, 1, 1, 0, 1$; $1, 0, 1, 1, 1, 0$; $1, 1, 0, 1, 0, 1$, respectively); we thus obtain the values 3, 5, and 6.

33. A plane P slices through a cube of volume 1 with a cross-section in the shape of a regular hexagon. This cube also has an inscribed sphere, whose intersection with P is a circle. What is the area of the region inside the regular hexagon but outside the circle?

Solution: $\boxed{(3\sqrt{3} - \pi)/4}$

One can show that the hexagon must have as its vertices the midpoints of six edges of the cube, as illustrated; for example, this readily follows from the fact that opposite sides of the hexagons and the medians between them are parallel. We then conclude that the side of the hexagon is $\sqrt{2}/2$ (since it cuts off an isosceles triangle of leg $1/2$ from each face), so the area is $(3/2)(\sqrt{2}/2)^2(\sqrt{3}) = 3\sqrt{3}/4$. Also, the plane passes through the center of the sphere by symmetry, so it cuts out a cross section of radius $1/2$, whose area (which is contained entirely inside the hexagon) is then $\pi/4$. The sought area is thus $(3\sqrt{3} - \pi)/4$.



34. Find the number of 20-tuples of integers $x_1, \ldots, x_{10}, y_1, \ldots, y_{10}$ with the following properties:

- $1 \le x_i \le 10$ and $1 \le y_i \le 10$ for each i;

- $x_i \leq x_{i+1}$ for $i = 1, \ldots, 9$;
- if $x_i = x_{i+1}$, then $y_i \leq y_{i+1}$.

Solution: $\boxed{\binom{109}{10}}$

By setting $z_i = 10x_i + y_i$, we see that the problem is equivalent to choosing a nondecreasing sequence of numbers $z_1, z_2, \ldots, z_{10}$ from the values $11, 12, \ldots, 110$. Making a further substitution by setting $w_i = z_i - 11 + i$, we see that the problem is equivalent to choosing a strictly increasing sequence of numbers $w_1, \ldots, w_{10}$ from among the values $1, 2, \ldots, 109$. There are $\binom{109}{10}$ ways to do this.

35. There are eleven positive integers n such that there exists a convex polygon with n sides whose angles, in degrees, are unequal integers that are in arithmetic progression. Find the sum of these values of n.

Solution: $\boxed{106}$

The sum of the angles of an n-gon is $(n-2)180$, so the average angle measure is $(n-2)180/n$. The common difference in this arithmetic progression is at least 1, so the difference between the largest and smallest angles is at least $n-1$. So the largest angle is at least $(n-1)/2 + (n-2)180/n$. Since the polygon is convex, this quantity is no larger than 179: $(n-1)/2 - 360/n \leq -1$, so that $360/n - n/2 \geq 1/2$. Multiplying by $2n$ gives $720 - n^2 \geq n$. So $n(n+1) \leq 720$, which forces $n \leq 26$. Of course, since the common difference is an integer, and the angle measures are integers, $(n-2)180/n$ must be an integer or a half integer, so $(n-2)360/n = 360 - 720/n$ is an integer, and then $720/n$ must be an integer. This leaves only $n = 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24$ as possibilities. When n is even, $(n-2)180/n$ is not an angle of the polygon, but the mean of the two middle angles. So the common difference is at least 2 when $(n-2)180/n$ is an integer. For $n = 20$, the middle angle is 162, so the largest angle is at least $162 + 38/2 = 181$, since 38 is no larger than the difference between the smallest and largest angles. For $n = 24$, the middle angle is 165, again leading to a contradiction. So no solution exists for $n = 20, 24$. All of the others possess solutions:

n	angles
3	$59, 60, 61$
4	$87, 89, 91, 93$
5	$106, 107, 108, 109, 110$
6	$115, 117, 119, 121, 123, 125$
8	$128, 130, 132, 134, 136, 138, 140, 142$
9	$136, \ldots, 144$
10	$135, 137, 139, \ldots, 153$
12	$139, 141, 143, \ldots, 161$
15	$149, 150, \ldots, 163$
16	$150, 151, \ldots, 165$
18	$143, 145, \ldots, 177$

(These solutions are quite easy to construct.) The desired value is then $3+4+5+6+8+9+10+12+15+16+18 = 106$.

36. For a string of P's and Q's, the *value* is defined to be the product of the positions of the P's. For example, the string $PPQPQQ$ has value $1 \cdot 2 \cdot 4 = 8$.

Also, a string is called *antipalindromic* if writing it backwards, then turning all the P's into Q's and vice versa, produces the original string. For example, $PPQPQQ$ is antipalindromic.

There are 2^{1002} antipalindromic strings of length 2004. Find the sum of the reciprocals of their values.

Solution: $\boxed{2005^{1002}/2004!}$

Consider the product

$$\left(\frac{1}{1}+\frac{1}{2004}\right)\left(\frac{1}{2}+\frac{1}{2003}\right)\left(\frac{1}{3}+\frac{1}{2002}\right)\cdots\left(\frac{1}{1002}+\frac{1}{1003}\right).$$

This product expands to 2^{1002} terms, and each term gives the reciprocal of the value of a corresponding antipalindromic string of P's and Q's as follows: if we choose the term $1/n$ for the nth factor, then our string has a P in position n and Q in position $2005-n$; if we choose the term $1/(2005-n)$, then we get a Q in position n and P in position $2005-n$. Conversely, each antipalindromic string has its value represented by exactly one of our 2^{1002} terms. So the value of the product is the number we are looking for. But when we simplify this product, the nth factor becomes $1/n + 1/(2005-n) = 2005/n(2005-n)$. Multiplying these together, we get 1002 factors of 2005 in the numerator and each integer from 1 to 2004 exactly once in the denominator, for a total of $2005^{1002}/2004!$.

37. Simplify $\prod_{k=1}^{2004} \sin(2\pi k/4009)$.

Solution: $\boxed{\frac{\sqrt{4009}}{2^{2004}}}$

Let $\zeta = e^{2\pi i/4009}$ so that $\sin(2\pi k/4009) = \frac{\zeta^k - \zeta^{-k}}{2i}$ and $x^{4009} - 1 = \prod_{k=0}^{4008}(x - \zeta^k)$. Hence $1 + x + \cdots + x^{4008} = \prod_{k=1}^{4008}(x - \zeta^k)$. Comparing constant coefficients gives $\prod_{k=1}^{4008} \zeta^k = 1$, setting $x = 1$ gives $\prod_{k=1}^{4008}(1-\zeta^k) = 4009$, and setting $x = -1$ gives $\prod_{k=1}^{4008}(1+\zeta^k) = 1$. Now, note that $\sin(2\pi(4009-k)/4009) = -\sin(2\pi k/4009)$, so

$$\begin{aligned}
\left(\prod_{k=1}^{2004} \sin(2\pi k/4009)\right)^2 &= (-1)^{2004} \prod_{k=1}^{4008} \sin(2\pi k/4009) \\
&= \prod_{k=1}^{4008} \frac{\zeta^k - \zeta^{-k}}{2i} \\
&= \frac{1}{(2i)^{4008}} \prod_{k=1}^{4008} \frac{\zeta^{2k}-1}{\zeta^k} \\
&= \frac{1}{2^{4008}} \prod_{k=1}^{4008} (\zeta^{2k}-1) \\
&= \frac{1}{2^{4008}} \prod_{k=1}^{4008} (\zeta^k-1)(\zeta^k+1) \\
&= \frac{4009 \cdot 1}{2^{4008}}.
\end{aligned}$$

However, $\sin(x)$ is nonnegative on the interval $[0, \pi]$, so our product is positive. Hence it is $\frac{\sqrt{4009}}{2^{2004}}$.

38. Let $S = \{p_1 p_2 \cdots p_n \mid p_1, p_2, \ldots, p_n$ are distinct primes and $p_1, \ldots, p_n < 30\}$. Assume 1 is in S. Let a_1 be an element of S. We define, for all positive integers n:

$$a_{n+1} = a_n/(n+1) \quad \text{if } a_n \text{ is divisible by } n+1;$$

$$a_{n+1} = (n+2)a_n \quad \text{if } a_n \text{ is not divisible by } n+1.$$

How many distinct possible values of a_1 are there such that $a_j = a_1$ for infinitely many j's?

Solution: $\boxed{512}$

If a_1 is odd, then we can see by induction that $a_j = (j+1)a_1$ when j is even and $a_j = a_1$ when j is odd (using the fact that no even j can divide a_1). So we have infinitely many j's for which $a_j = a_1$.

If $a_1 > 2$ is even, then a_2 is odd, since $a_2 = a_1/2$, and a_1 may have only one factor of 2. Now, in general, let $p = \min(\{p_1, \ldots, p_n\} \setminus \{2\})$. Suppose $1 < j < p$. By induction, we have $a_j = (j+1)a_1/2$ when j is odd, and $a_j = a_1/2$ when j is even. So $a_i \neq a_1$ for all $1 < j < p$. It follows that $a_p = a_1/2p$. Then, again using induction, we get for all nonnegative integers k that $a_{p+k} = a_p$ if k is even, and $a_{p+k} = (p+k+1)a_p$ if k is odd. Clearly, $a_p \neq a_1$ and $p+k+1 \neq 2p$ when k is odd (the left side is odd, and the right side even). It follows that $a_j = a_1$ for no $j > 1$. Finally, when $a_1 = 2$, we can check inductively that $a_j = j+1$ for j odd and $a_j = 1$ for j even.

So our answer is just the number of odd elements in S. There are 9 odd prime numbers smaller than 30, so the answer is $2^9 = 512$.

39. You want to arrange the numbers $1, 2, 3, \ldots, 25$ in a sequence with the following property: if n is divisible by m, then the nth number is divisible by the mth number. How many such sequences are there?

Solution: $\boxed{24}$

Let the rearranged numbers be $a_1, \ldots, a_{25}$. The number of pairs (n, m) with $n \mid m$ must equal the number of pairs with $a_n \mid a_m$, but since each pair of the former type is also of the latter type, the converse must be true as well. Thus, $n \mid m$ if and only if $a_n \mid a_m$. Now for each $n = 1, 2, \ldots, 6$, the number of values divisible by n uniquely determines n, so $n = a_n$. Similarly, $7, 8$ must either be kept fixed by the rearrangement or interchanged, because they are the only values that divide exactly 2 other numbers in the sequence; since 7 is prime and 8 is not, we conclude they are kept fixed. Then we can easily check by induction that $n = a_n$ for all larger composite numbers $n \leq 25$ (by using $m = a_m$ for all proper factors m of n) and $n = 11$ (because it is the only prime that divides exactly 1 other number). So we have only the primes $n = 13, 17, 19, 23$ left to rearrange, and it is easily seen that these can be permuted arbitrarily, leaving 4! possible orderings altogether.

40. You would like to provide airline service to the 10 cities in the nation of Schizophrenia, by instituting a certain number of two-way routes between cities. Unfortunately, the government is about to divide Schizophrenia into two warring countries of five cities

each, and you don't know which cities will be in each new country. All airplane service between the two new countries will be discontinued. However, you want to make sure that you set up your routes so that, for any two cities in the same new country, it will be possible to get from one city to the other (without leaving the country).

What is the minimum number of routes you must set up to be assured of doing this, no matter how the government divides up the country?

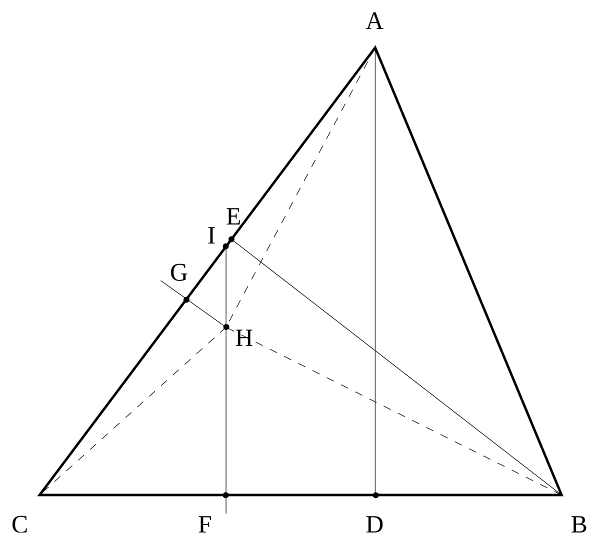
Solution: $\boxed{30}$

Each city C must be directly connected to at least 6 other cities, since otherwise the government could put C in one country and all its connecting cities in the other country, and there would be no way out of C. This means that we have 6 routes for each of 10 cities, counted twice (since each route has two endpoints) $\Rightarrow 6 \cdot 10/2 = 30$ routes. On the other hand, this is enough: picture the cities arranged around a circle, and each city connected to its 3 closest neighbors in either direction. Then if C and D are in the same country but mutually inaccessible, this means that on each arc of the circle between C and D, there must be (at least) three consecutive cities in the other country. Then this second country would have 6 cities, which is impossible. So our arrangement achieves the goal with 30 routes.

41. A tetrahedron has all its faces triangles with sides 13, 14, 15. What is its volume?

Solution: $\boxed{42\sqrt{55}}$

Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. Let AD, BE be altitudes. Then $BD = 5, CD = 9$. (If you don't already know this, it can be deduced from the Pythagorean Theorem: $CD^2 - BD^2 = (CD^2 + AD^2) - (BD^2 + AD^2) = AC^2 - AB^2 = 56$, while $CD + BD = BC = 14$, giving $CD - BD = 56/14 = 4$, and now solve the linear system.) Also, $AD = \sqrt{AB^2 - BD^2} = 12$. Similar reasoning gives $AE = 33/5$, $EC = 42/5$.



Now let F be the point on BC such that $CF = BD = 5$, and let G be on AC such that $CG = AE = 33/5$. Imagine placing face ABC flat on the table, and letting X be a point in space with $CX = 13, BX = 14$. By mentally rotating triangle BCX

about line BC, we can see that X lies on the plane perpendicular to BC through F. In particular, this holds if X is the fourth vertex of our tetrahedron $ABCX$. Similarly, X lies on the plane perpendicular to AC through G. Let the mutual intersection of these two planes and plane ABC be H. Then XH is the altitude of the tetrahedron.

To find XH, extend FH to meet AC at I. Then $\triangle CFI \sim \triangle CDA$, a 3-4-5 triangle, so $FI = CF \cdot 4/3 = 20/3$, and $CI = CF \cdot 5/3 = 25/3$. Then $IG = CI - CG = 26/15$, and $HI = IG \cdot 5/4 = 13/6$. This leads to $HF = FI - HI = 9/2$, and finally $XH = \sqrt{XF^2 - HF^2} = \sqrt{AD^2 - HF^2} = 3\sqrt{55}/2$.

Now $XABC$ is a tetrahedron whose base $\triangle ABC$ has area $AD \cdot BC/2 = 12 \cdot 14/2 = 84$, and whose height XH is $3\sqrt{55}/2$, so its volume is $(84)(3\sqrt{55}/2)/3 = 42\sqrt{55}$.

42. S is a set of complex numbers such that if $u, v \in S$, then $uv \in S$ and $u^2 + v^2 \in S$. Suppose that the number N of elements of S with absolute value at most 1 is finite. What is the largest possible value of N?

Solution: $\boxed{13}$

First, if S contained some $u \neq 0$ with absolute value < 1, then (by the first condition) every power of u would be in S, and S would contain infinitely many different numbers of absolute value < 1. This is a contradiction. Now suppose S contains some number u of absolute value 1 and argument θ. If θ is not an integer multiple of $\pi/6$, then u has some power v whose argument lies strictly between $\theta + \pi/3$ and $\theta + \pi/2$. Then $u^2 + v^2 = u^2(1 + (v/u)^2)$ has absolute value between 0 and 1, since $(v/u)^2$ lies on the unit circle with angle strictly between $2\pi/3$ and π. But $u^2 + v^2 \in S$, so this is a contradiction.

This shows that the only possible elements of S with absolute value ≤ 1 are 0 and the points on the unit circle whose arguments are multiples of $\pi/6$, giving $N \leq 1+12 = 13$. To show that $N = 13$ is attainable, we need to show that there exists a possible set S containing all these points. Let T be the set of all numbers of the form $a + b\omega$, where a, b are integers are ω is a complex cube root of 1. Since $\omega^2 = -1 - \omega$, T is closed under multiplication and addition. Then, if we let S be the set of numbers u such that $u^2 \in T$, S has the required properties, and it contains the 13 complex numbers specified, so we're in business.

43. Write down an integer from 0 to 20 inclusive. This problem will be scored as follows: if N is the second-largest number from among the responses submitted, then each team that submits N gets N points, and everyone else gets zero. (If every team picks the same number then nobody gets any points.)

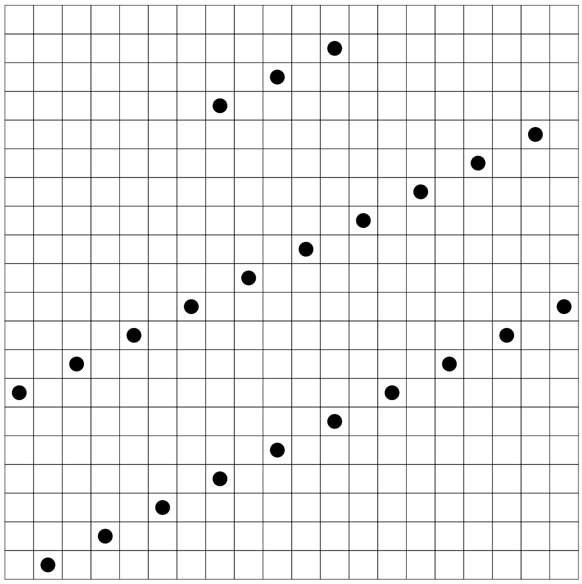
Solution: The only Nash equilibria of this game (where each team plays its best possible move given the other teams' choices) are fairly degenerate: every team but one plays 1, and the remaining team is more likely to choose 2 than any higher number. Of course, we cannot assume perfectly rational play in reality — nor are the utility functions the same, since the goal is to score higher than other teams, not to maximize one's own expected number of points. It will be interesting to see what the submissions are.

44. Shown on your answer sheet is a 20×20 grid. Place as many queens as you can so that each of them attacks at most one other queen. (A queen is a chess piece that can

move any number of squares horizontally, vertically, or diagonally.) It's not very hard to get 20 queens, so you get no points for that, but you get 5 points for each further queen beyond 20. You can mark the grid by placing a dot in each square that contains a queen.

Solution: An elementary argument shows there cannot be more than 26 queens: we cannot have more than 2 in a row or column (or else the middle queen would attack the other two), so if we had 27 queens, there would be at least 7 columns with more than one queen and thus at most 13 queens that are alone in their respective columns. Similarly, there would be at most 13 queens that are alone in their respective rows. This leaves $27 - 13 - 13 = 1$ queen who is not alone in her row or column, and she therefore attacks two other queens, contradiction.

Of course, this is not a very strong argument since it makes no use of the diagonals. The best possible number of queens is not known to us; the following construction gives 23:



45. A *binary string of length n* is a sequence of n digits, each of which is 0 or 1. The *distance* between two binary strings of the same length is the number of positions in which they disagree; for example, the distance between the strings 01101011 and 00101110 is 3 since they differ in the second, sixth, and eighth positions.

Find as many binary strings of length 8 as you can, such that the distance between any two of them is at least 3. You get one point per string.

Solution: The maximum possible number of such strings is 20. An example of a set attaining this bound is

00000000	00110101
11001010	10011110
11100001	01101011
11010100	01100110
10111001	10010011
01111100	11001101
00111010	10101100
01010111	11110010
00001111	01011001
10100111	11111111

This example is taken from page 57 of F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error Correcting Codes* (New York: Elsevier Publishing, 1977). The proof that 20 is the best possible is elementary but too long to reproduce here; see pages 537–541 of MacWilliams and Sloane for details.

In general, a set of M strings of length n such that any two have a distance of at least d is called an (n, M, d)*-code*. These objects are of basic importance in coding theory, which studies how to transmit information through a channel with a known error rate. For example, since the code given above has minimum distance 3, I can transmit to you a message consisting of strings in this code, and even if there is a possible error rate of one digit in each string, you will still be able to determine the intended message uniquely.