Harvard-MIT Mathematics Tournament

February 28, 2004

Team Round — Solutions

A Build-It-Yourself Table

An infinite table of nonnegative integers is constructed as follows: in the top row, some number is 1 and all other numbers are 0's; in each subsequent row, every number is the sum of some two of the three closest numbers in the preceding row. An example of such a table is shown below.

$\cdots$	0	0	0	0	1	0	0	0	0	$\cdots$
$\cdots$	0	0	0	0	1	1	0	0	0	$\cdots$
$\cdots$	0	0	0	1	1	2	1	0	0	$\cdots$
$\cdots$	0	0	1	1	3	3	2	0	0	$\cdots$
$\cdots$	0	1	2	4	4	6	3	2	0	$\cdots$
$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The top row (with the one 1) is called row 0; the next row is row 1; the next row is row 2, and so forth.

Note that the following problems require you to prove the statements for *every* table that can be constructed by the process described above, not just for the example shown.

1. Show that any number in row n (for $n > 0$) is at most 2^{n-1}.

   Solution: We use induction on n. It is clear that any number in row 1 is at most $1 = 2^0$. Now, if every number in row n is at most 2^{n-1}, then every number in row $n+1$ is the sum of two numbers in row n and so is at most $2^{n-1} + 2^{n-1} = 2^n$. This gives the induction step, and the result follows.

2. What is the earliest row in which the number 2004 may appear?

   Solution:

$\cdots$	0	0	1	0	0	$\cdots$
$\cdots$	0	0	1	1	0	$\cdots$
$\cdots$	0	1	2	2	0	$\cdots$
$\cdots$	0	3	4	4	0	$\cdots$
$\cdots$	0	7	8	8	0	$\cdots$
$\cdots$	0	15	16	16	0	$\cdots$
$\cdots$	0	31	31	32	0	$\cdots$
$\cdots$	0	62	63	63	0	$\cdots$
$\cdots$	0	125	125	126	0	$\cdots$
$\cdots$	0	250	251	251	0	$\cdots$
$\cdots$	0	501	501	502	0	$\cdots$
$\cdots$	0	1002	1002	1003	0	$\cdots$
$\cdots$	0	2004	2004	2005	0	$\cdots$

   By the previous problem, it cannot appear before row 12. By starting off the table as shown above, we see that row 12 is possible, so this is the answer.

3. Let
$$S(n,r) = \binom{n-1}{r-1} + \binom{n-1}{r} + \binom{n-1}{r+1} + \cdots + \binom{n-1}{n-1}$$
for all $n, r > 0$, and in particular $S(n,r) = 0$ if $r > n > 0$. Prove that the number in row n of the table, r columns to the left of the 1 in the top row, is at most $S(n,r)$. (**Hint**: First prove that $S(n-1,r-1) + S(n-1,r) = S(n,r)$.)

   Solution: First, we prove the statement in the hint: adding the ith term of the sum for $S(n-1,r-1)$ to the ith term for $S(n-1,r)$, for each i, we get that $S(n-1,r-1) + S(n-1,r)$ equals
$$\left(\binom{n-2}{r-2} + \binom{n-2}{r-1}\right) + \left(\binom{n-2}{r-1} + \binom{n-2}{r}\right) + \cdots$$
$$\cdots + \left(\binom{n-2}{n-3} + \binom{n-2}{n-2}\right) + \binom{n-2}{n-2}$$
$$= \binom{n-1}{r-1} + \binom{n-1}{r} + \cdots + \binom{n-1}{n-1} = S(n,r).$$

   Now we can prove the main statement by induction on n. The base case $n = 1$ is clear. If the statement holds for $n-1$, then first suppose $r > 1$. Then the number in row n, r columns to the left, is the sum of two of the three numbers above it, which, by the induction hypothesis, are at most $S(n-1,r-1), S(n-1,r), S(n-1,r+1)$ respectively. Since the first two of these are greater than the last (because the summation formula gives $S(n-1,r-1) = S(n-1,r) + \binom{n-1}{r-2}$ and $S(n-1,r) = S(n-1,r+1) + \binom{n-1}{r-1}$), we have an upper bound of $S(n-1,r-1) + S(n-1,r) = S(n,r)$ by the above. So the result follows by induction. Finally, in the case $r = 1$, the quantity in question is just 2^{n-1}, and the result holds by Problem 1.

4. Show that the sum of all the numbers in row n is at most $(n+2)2^{n-1}$.

   Solution: The previous problem gives an upper bound on the number located r columns to the left of the initial 1; adding over all $r = 1, 2, \ldots, n$ gives
$$\sum_{s=0}^{n-1} (s+1)\binom{n-1}{s}$$
since the term $\binom{n-1}{s}$ occurs for the $s+1$ values $r = 1, 2, \ldots, s+1$. But this sum equals $(n+1)2^{n-2}$. For example, add the sum to itself and reverse the terms of the second sum to get
$$\sum_{s=0}^{n-1} (s+1)\binom{n-1}{s} + \sum_{s=0}^{n-1} ([n-1-s]+1)\binom{n-1}{n-1-s}$$
$$= \sum_{s=0}^{n-1} (n+1)\binom{n-1}{s} = (n+1)\sum_{s=0}^{n-1}\binom{n-1}{s} = (n+1)2^{n-1},$$
and our original sum is half of this.

   So the sum of the terms in row n to the left of the central column is at most $(n+1)2^{n-2}$. Similarly, the sum of the terms to the right of the central column is at most $(n+1)2^{n-2}$. Adding these together, plus the upper bound of 2^{n-1} for the central number (Problem 1), gives our upper bound of $(n+2)2^{n-1}$ for the sum of all the numbers in the row.

A pair of successive numbers in the same row is called a *switch pair* if one number in the pair is even and the other is odd.

5. Prove that the number of switch pairs in row n is at most twice the number of odd numbers in row n.

   Solution: Each switch pair contains an odd number, and each odd number can belong to at most two switch pairs (since it has only two neighbors).

6. Prove that the number of odd numbers in row n is at most twice the number of switch pairs in row $n-1$.

   Solution: Each odd number in row n is the sum of two of the three numbers above it in row $n-1$; these three numbers cannot all have the same parity (or else any sum of two of them would be even), so somewhere among them is a switch pair. Since each switch pair in row $n-1$ can contribute to at most two odd numbers in row n in this manner (namely, the two numbers immediately below the members of the pair), the result follows.

7. Prove that the number of switch pairs in row n is at most twice the number of switch pairs in row $n-1$.

   Solution: If we go sufficiently far to the left or right in row n, we get to zeroes. Therefore, row n consists of a finite number of "odd blocks" of the form

   $$EOOO\cdots OE$$

   (where E represents an even number and O an odd number), which are separated by even numbers, except that one even number may simultaneously be an endpoint of two odd blocks. Each odd block contributes two switch pairs to row n, so it is enough to show that each odd block has a switch pair somewhere above it in row $n-1$. But the odd block consists of at least one O between two E's, making for at least three terms. If there were no switch pairs above the block, then in particular, the first three terms immediately above it would be all odd or all even, and then the second term in our block would have to be even, contradicting the assumption that it was O. This proves the result.

Written In The Stars

Suppose S is a finite set with a binary operation $\star$ — that is, for any elements a, b of S, there is defined an element $a \star b$ of S. It is given that $(a \star b) \star (a \star b) = b \star a$ for all $a, b \in S$.

8. Prove that $a \star b = b \star a$ for all $a, b \in S$.

   Solution: We have

   $$\begin{aligned} a \star b &= (b \star a) \star (b \star a) \\ &= ([a \star b] \star [a \star b]) \star ([a \star b] \star [a \star b]) \\ &= [a \star b] \star [a \star b] \\ &= b \star a. \end{aligned}$$

Let T be the set of elements of the form $a \star a$ for $a \in S$.

9. If b is any element of T, prove that $b \star b = b$.

   Solution: If $b \in T$, then $b = a \star a$ for some a, so $b \star b = (a \star a) \star (a \star a) = a \star a$ (by the given property) $= b$.

Now suppose further that $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in S$. (Thus we can write an expression like $a \star b \star c \star d$ without ambiguity.)

10. Let a be an element of T. Let the *image* of a be the set of all elements of T that can be represented as $a \star b$ for some $b \in T$. Prove that if c is in the image of a, then $a \star c = c$.

    Solution: Write $c = a \star b$, and then $a \star c = a \star a \star b = a \star b$ (since $a \star a = a$) $= c$.

11. Prove that there exists an element $a \in T$ such that the equation $a \star b = a$ holds for all $b \in T$.

    Solution: Choose a whose image contains as few elements as possible — we know we can do this, since T, being a subset of S, is finite. We claim that this a works. Indeed, suppose c is in the image of a. Then, for any d in the image of c, $a \star (c \star d) = (a \star c) \star d = c \star d = d$, so d is also in the image of a. So the image of c is contained in the image of a. But a was chosen to have image as small as possible, so the two images must coincide. In particular, $a \star a = a$ is in the image of c. So

    $$a = c \star a = a \star c = c.$$

    This argument shows that a is the only element of the image of a, which gives what we wanted.

    Alternative Solution: This can also be solved without using Problem 10: The product of any two elements of T is also in T, since commutativity and associativity give $(b \star b) \star (c \star c) = (b \star c) \star (b \star c)$ for $b, c \in S$. Then let $a_1, a_2, \ldots, a_n$ be all the elements of T, and put $a = a_1 \star a_2 \star \cdots \star a_n$; this value does not depend on the ordering of the elements. If $b \in T$, then $a = c \star b$, where c is the $\star$-product of all elements of T different from b, and consequently $a \star b = (c \star b) \star b = c \star (b \star b) = c \star b = a$.

12. Prove that there exists an element $a \in S$ such that the equation $a \star b = a$ holds for all $b \in S$.

    Solution: The same a as in the previous problem will do the trick. Indeed, for any $b \in S$, we have

    $$a \star b = (b \star a) \star (b \star a) = (a \star a) \star (b \star b)$$

    (we have used commutativity and associativity). But $a \star a = a$, and $b \star b \in T$, so this expression equals $a \star (b \star b) = a$, as required.

Sigma City

13. Let n be a positive odd integer. Prove that

    $$\lfloor \log_2 n \rfloor + \lfloor \log_2(n/3) \rfloor + \lfloor \log_2(n/5) \rfloor + \lfloor \log_2(n/7) \rfloor + \cdots + \lfloor \log_2(n/n) \rfloor = (n-1)/2.$$

Solution: Note that $\lfloor \log_2 k \rfloor$ is the cardinality of the set $\{2, 4, 8, \ldots, 2^{\lfloor \log_2 k \rfloor}\}$, i.e., the number of powers of 2 that are even and are at most k. Then $\lfloor \log_2(n/k) \rfloor$ is the number of even powers of 2 that are at most n/k, or equivalently (multiplying each such number by k) the number of positive even numbers $\leq n$ whose greatest odd divisor is k. Summing over all odd k, we get the number of even numbers $\leq n$, which is just $(n-1)/2$.

Let $\sigma(n)$ denote the sum of the (positive) divisors of n, including 1 and n itself.

14. Prove that
$$\sigma(1) + \sigma(2) + \sigma(3) + \cdots + \sigma(n) \leq n^2$$
for every positive integer n.

Solution: The ith term on the left is the sum of all d dividing i. If we write this sum out explicitly, then each term $d = 1, 2, \ldots, n$ appears $\lfloor n/d \rfloor$ times — once for each multiple of d that is $\leq n$. Thus, the sum equals
$$\begin{aligned} \lfloor n/1 \rfloor + 2\lfloor n/2 \rfloor + 3\lfloor n/3 \rfloor + \cdots + n\lfloor n/n \rfloor &\leq n/1 + 2n/2 + 3n/3 + \cdots + n/n \\ &= n + n + \cdots + n \\ &= n^2. \end{aligned}$$

15. Prove that
$$\frac{\sigma(1)}{1} + \frac{\sigma(2)}{2} + \frac{\sigma(3)}{3} + \cdots + \frac{\sigma(n)}{n} \leq 2n$$
for every positive integer n.

Solution: This is similar to the previous solution. If d is a divisor of i, then so is i/d, and $(i/d)/i = 1/d$. Summing over all d, we see that $\sigma(i)/i$ is the sum of the reciprocals of the divisors of i, for each positive integer i. So, summing over all i from 1 to n, we get the value $1/d$ appearing $\lfloor n/d \rfloor$ times, once for each multiple of d that is at most n. In particular, the sum is
$$\frac{1}{1}\left\lfloor \frac{n}{1} \right\rfloor + \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{3}\left\lfloor \frac{n}{3} \right\rfloor + \cdots + \frac{1}{n}\left\lfloor \frac{n}{n} \right\rfloor < \frac{n}{1^2} + \frac{n}{2^2} + \cdots + \frac{n}{n^2}.$$
So now all we need is $1/1^2 + 1/2^2 + \cdots + 1/n^2 < 2$. This can be obtained from the classic formula $1/1^2 + 1/2^2 + \cdots = \pi^2/6$, or from the more elementary estimate
$$\begin{aligned} 1/2^2 + 1/3^2 + \cdots + 1/n^2 &< 1/(1 \cdot 2) + 1/(2 \cdot 3) + \cdots + 1/\big((n-1) \cdot n\big) \\ &= (1/1 - 1/2) + (1/2 - 1/3) + \cdots + \big(1/(n-1) - 1/n\big) \\ &= 1 - 1/n \\ &< 1. \end{aligned}$$

16. Now suppose again that n is odd. Prove that
$$\sigma(1)\lfloor \log_2 n \rfloor + \sigma(3)\lfloor \log_2(n/3) \rfloor + \sigma(5)\lfloor \log_2(n/5) \rfloor + \cdots + \sigma(n)\lfloor \log_2(n/n) \rfloor < n^2/8.$$

Solution: The term $\sigma(i)\lfloor \log_2(n/i) \rfloor$ is the sum of the divisors of i times the number of even numbers $\leq n$ whose greatest odd divisor is i. Thus, summing over all odd i, we get the sum of d over all pairs (d, j), where $j < n$ is even and d is an odd divisor of j. Each odd number d then appears $\lfloor n/2d \rfloor$ times, since this is the number of even numbers $< n$ that have d as a divisor. So the sum equals

$$\lfloor n/2 \rfloor + 3 \lfloor n/6 \rfloor + 5 \lfloor n/10 \rfloor + \cdots + n \lfloor n/2n \rfloor \\ \leq (n-1)/2 + 3(n-1)/6 + \cdots + m(n-1)/2m,$$

where m is the greatest odd integer less than $n/2$. (We can ignore the terms $d \lfloor n/2d \rfloor$ for $d > m$ because these floors are zero.) This expression equals

$$(n-1)/2 + (n-1)/2 + \cdots + (n-1)/2 = (n-1)(m+1)/4 \leq (n-1)(n+1)/8,$$

which is less than $n^2/8$, as required.