Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Geometry Subject Test — Solutions

1. The volume of a cube (in cubic inches) plus three times the total length of its edges (in inches) is equal to twice its surface area (in square inches). How many inches long is its long diagonal?

   Solution: $\boxed{6\sqrt{3}}$

   If the side length of the cube is s inches, then the condition implies $s^3 + 3 \cdot 12s = 2 \cdot 6s^2$, or $s(s^2 - 12s + 36) = s(s-6)^2 = 0$. Therefore $s = 6$, and the long diagonal has length $s\sqrt{3} = 6\sqrt{3}$.

2. Let $ABCD$ be a regular tetrahedron with side length 2. The plane parallel to edges AB and CD and lying halfway between them cuts $ABCD$ into two pieces. Find the surface area of one of these pieces.

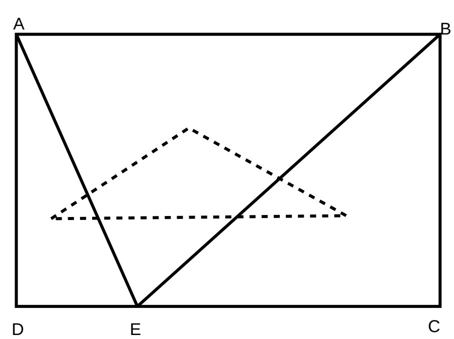
   Solution: $\boxed{1 + 2\sqrt{3}}$

   The plane intersects each face of the tetrahedron in a midline of the face; by symmetry it follows that the intersection of the plane with the tetrahedron is a square of side length 1. The surface area of each piece is half the total surface area of the tetrahedron plus the area of the square, that is, $\frac{1}{2} \cdot 4 \cdot \frac{2^2\sqrt{3}}{4} + 1 = 1 + 2\sqrt{3}$.

3. Let $ABCD$ be a rectangle with area 1, and let E lie on side CD. What is the area of the triangle formed by the centroids of triangles ABE, BCE, and ADE?

   Solution: $\boxed{1/9}$

   Let the centroids of ABE, BCE, and ADE be denoted by X, Y, and Z, respectively. Let $d(P, QR)$ denote the distance from P to line QR. Since the centroid lies two-thirds of the distance from each vertex to the midpoint of the opposite edge, $d(X, AB) = d(Y, CD) = d(Z, CD) = \frac{1}{3}BC$, so YZ is parallel to CD and $d(X, YZ) = BC - \frac{2}{3}BC = \frac{1}{3}BC$. Likewise, $d(Z, AD) = \frac{1}{3}DE$ and $d(Y, BC) = \frac{1}{3}CE$, so that since YZ is perpendicular to AD and BC, we have that $YZ = CD - \frac{1}{3}(DE + CE) = \frac{2}{3}CD$. Therefore, the area of XYZ is $\frac{1}{2}(\frac{1}{3}BC)(\frac{2}{3}CD) = \frac{1}{9}BC \cdot CD = \frac{1}{9}$.



4. Let XYZ be a triangle with $\angle X = 60°$ and $\angle Y = 45°$. A circle with center P passes through points A and B on side XY, C and D on side YZ, and E and F on side ZX. Suppose $AB = CD = EF$. Find $\angle XPY$ in degrees.

Solution: $\boxed{255/2}$

Since PAB, PCD, and PEF are all isosceles triangles with equal legs and equal bases, they are congruent. It follows that the heights of each are the same, so that P is equidistant from the sides of XYZ. Therefore, P is the incenter and therefore lies on the angle bisectors of XYZ. Thus $\angle YXP = \frac{1}{2}\angle YXZ = 30^\circ$ and $\angle PYX = \frac{1}{2}\angle ZYX = \frac{45}{2}^\circ$. Therefore $\angle XPY = 180^\circ - 30^\circ - \frac{45}{2}^\circ = \frac{255}{2}^\circ$.

5. A cube with side length 2 is inscribed in a sphere. A second cube, with faces parallel to the first, is inscribed between the sphere and one face of the first cube. What is the length of a side of the smaller cube?

   Solution: $\boxed{2/3}$

   First note that the long diagonal of the cube has length $2\sqrt{3}$, so the radius of the sphere is $\sqrt{3}$. Let x be the side length of the smaller cube. Then the distance from the center of the sphere to the far face of the smaller cube is $1+x$, while the distance from the center of the far face to a vertex lying on the sphere is $\frac{x\sqrt{2}}{2}$. Therefore, the square of the radius is $3 = (1+x)^2 + \frac{x^2}{2}$, or $3x^2 + 4x - 4 = (3x-2)(x+2) = 0$, so $x = \frac{2}{3}$.

6. A triangular piece of paper of area 1 is folded along a line parallel to one of the sides and pressed flat. What is the minimum possible area of the resulting figure?

   Solution: $\boxed{2/3}$

   Let the triangle be denoted ABC, and suppose we fold parallel to BC. Let the distance from A to BC be h, and suppose we fold along a line at a distance of ch from A. We will assume that neither angle B nor C is obtuse, for the area of overlap will only be smaller if either is obtuse. If $c \leq \frac{1}{2}$, then A does not fold past the edge BC, so the overlap is a triangle similar to the original with height ch; the area of the figure is then $1 - c^2 \geq \frac{3}{4}$. Suppose $c > \frac{1}{2}$, so that A does fold past BC. Then the overlap is a trapezoid formed by taking a triangle of height ch similar to the original and removing a triangle of height $(2c-1)h$ similar to the original. The area of the resulting figure is thus $1 - c^2 + (2c-1)^2 = 3c^2 - 4c + 2$. This is minimized when $c = \frac{2}{3}$, when the area is $\frac{2}{3} < \frac{3}{4}$; the minimum possible area is therefore $\frac{2}{3}$.

7. Let $ABCD$ be a tetrahedron such that edges AB, AC, and AD are mutually perpendicular. Let the areas of triangles ABC, ACD, and ADB be denoted by x, y, and z, respectively. In terms of x, y, and z, find the area of triangle BCD.

   Solution: $\boxed{\sqrt{x^2+y^2+z^2}}$

   Place A, B, C, and D at $(0,0,0)$, $(b,0,0)$, $(0,c,0)$, and $(0,0,d)$ in Cartesian coordinate space, with b, c, and d positive. Then the plane through B, C, and D is given by the equation $\frac{x}{b} + \frac{y}{c} + \frac{z}{d} = 1$. The distance from the origin to this plane is then

   $$\frac{1}{\sqrt{\frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}}} = \frac{bcd}{\sqrt{b^2c^2 + c^2d^2 + d^2b^2}} = \frac{bcd}{2\sqrt{x^2+y^2+z^2}}.$$

   Then if the area of BCD is K, the volume of the tetrahedron is

   $$\frac{bcd}{6} = \frac{bcdK}{6\sqrt{x^2+y^2+z^2}},$$

implying $K = \sqrt{x^2 + y^2 + z^2}$.

Alternative Solution: The area of BCD is also half the length of the cross product of the vectors $\overrightarrow{BC} = (0, -c, d)$ and $\overrightarrow{BD} = (-b, 0, d)$. This cross product is $(-cd, -db, -bc) = -2(y, z, x)$, which has length $2\sqrt{x^2 + y^2 + z^2}$. Thus the area of BCD is $\sqrt{x^2 + y^2 + z^2}$.

8. Let T be a triangle with side lengths 26, 51, and 73. Let S be the set of points inside T which do not lie within a distance of 5 of any side of T. Find the area of S.

   Solution: $\boxed{135/28}$

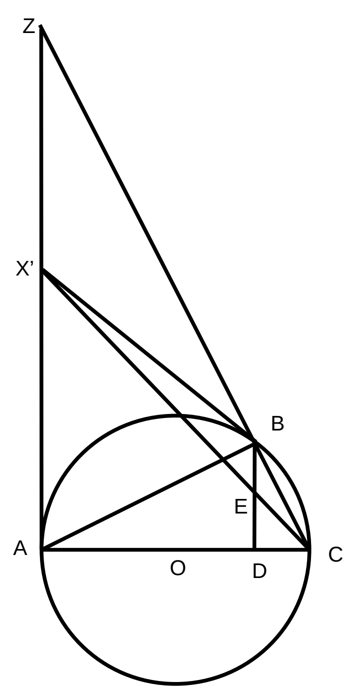
   Note that the sides of S are parallel to the sides of T, so S is a triangle similar to T. The semiperimeter of T is $s = \frac{1}{2}(26 + 51 + 73) = 75$. By Heron's formula, the area of T is $\sqrt{75 \cdot 49 \cdot 24 \cdot 2} = 420$. If r is the inradius of T, then the area of T is rs, so $r = 420/75 = 28/5$. It follows that the inradius of S is $r - 5 = 3/5$, and the ratio of similitude between S and T is $3/28$. Therefore, the area of S is $420 \cdot (3/28)^2 = 135/28$.

9. Let AC be a diameter of a circle ω of radius 1, and let D be the point on AC such that $CD = 1/5$. Let B be the point on ω such that DB is perpendicular to AC, and let E be the midpoint of DB. The line tangent to ω at B intersects line CE at the point X. Compute AX.

   Solution: $\boxed{3}$

   We first show that AX is perpendicular to AC. Let the tangent to ω at A intersect CB at Z and CE at X'. Since ZA is parallel to BD and $BE = ED$, $ZX' = X'A$. Therefore, X' is the midpoint of the hypotenuse of the right triangle ABZ, so it is also its circumcenter. Thus $X'A = X'B$, and since $X'A$ is tangent to ω and B lies on ω, we must have that $X'B$ is tangent to ω, so $X = X'$.

   Let O be the center of ω. Then $OD = \frac{4}{5}$, so $BD = \frac{3}{5}$ and $DE = \frac{3}{10}$. Then $AX = DE \cdot \frac{AC}{DC} = \frac{3}{10} \cdot \frac{2}{1/5} = 3$.



10. Let AB be the diameter of a semicircle Γ. Two circles, ω_1 and ω_2, externally tangent to each other and internally tangent to Γ, are tangent to the line AB at P and Q, respectively, and to semicircular arc AB at C and D, respectively, with $AP < AQ$. Suppose F lies on Γ such that $\angle FQB = \angle CQA$ and that $\angle ABF = 80°$. Find $\angle PDQ$ in degrees.

Solution: $\boxed{35}$

Extend the semicircle centered at O to an entire circle ω, and let the reflection of F over AB be F'. Then CQF' is a straight line. Also, the homothety centered at C taking ω_1 into ω takes P to a point X on ω and AB to the parallel line tangent to ω at X. Therefore, X is the midpoint of semicircle AXB, and C, P, and X lie on a line. Similarly, D, Q, and X lie on a line. So,

$$45° = \angle XCB = \angle PCB = \angle PCQ + \angle QCB = \angle PCQ + 10°,$$

since $\angle QCB = \angle F'CB = \angle F'AB = \angle FAB = 90° - \angle ABF = 10°$. Thus $\angle PCQ = 35°$. We will show that $\angle PCQ = \angle PDQ$ to get that $\angle PDQ = 35°$.

Note that $\angle XPQ$ subtends the sum of arcs AC and BX, which is equal to arc XC. Therefore $\angle XPQ = \angle CDX$, so $CDQP$ is cyclic and $\angle PCQ = \angle PDQ$. The conclusion follows.