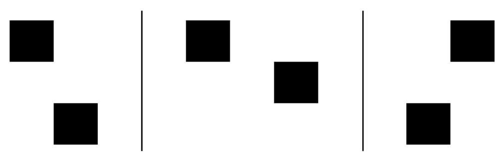
Harvard-MIT Mathematics Tournament

February 19, 2005

Team Round B — Solutions

Disconnected Domino Rally [**150**]

On an infinite checkerboard, the union of any two distinct unit squares is called a *(disconnected) domino*. A domino is said to be of *type* (a, b), with $a \leq b$ integers not both zero, if the centers of the two squares are separated by a distance of a in one orthogonal direction and b in the other. (For instance, an ordinary connected domino is of type $(0, 1)$, and a domino of type $(1, 2)$ contains two squares separated by a knight's move.)



Each of the three pairs of squares above forms a domino of type $(1, 2)$.

Two dominoes are said to be *congruent* if they are of the same type. A rectangle is said to be (a, b)-*tileable* if it can be partitioned into dominoes of type (a, b).

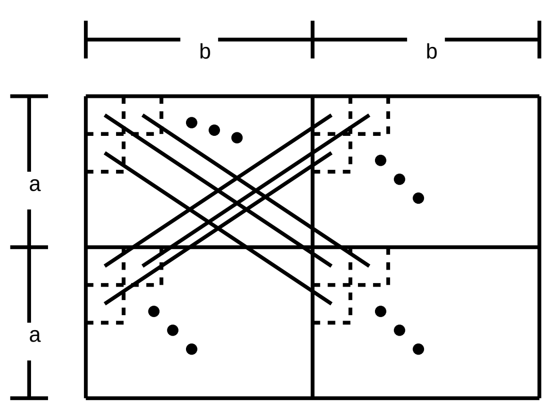
1. [**15**] Let $0 < m \leq n$ be integers. How many different (i.e., noncongruent) dominoes can be formed by choosing two squares of an $m \times n$ array?

   Solution: We must have $0 \leq a < m$, $0 \leq b < n$, $a \leq b$, and a and b not both 0. The number of pairs (a, b) with $b < a < m$ is $m(m-1)/2$, so the answer is

$$mn - \frac{m(m-1)}{2} - 1 = mn - \frac{m^2 - m + 2}{2}.$$

2. [**10**] What are the dimensions of the rectangle of smallest area that is (a, b)-tileable?

   Solution: If $a = 0$, then a $1 \times 2b$ rectangle is tileable in an obvious way. If $a > 0$, then a $2a \times 2b$ rectangle is tileable by dividing the rectangle into quarters and pairing each square with a square from the diagonally opposite quarter. The answer is therefore $\max\{1, 2a\} \times 2b$. Minimality of area follows from the next question.



3. **[20]** Prove that every (a,b)-tileable rectangle contains a rectangle of these dimensions.

   Solution: An $m \times n$ rectangle, $m \leq n$, does not contain a $\max\{1, 2a\} \times 2b$ rectangle if and only if $m < 2a$ or $n < 2b$. But if either of these is the case, then the square closest to the center of the rectangle cannot be paired with any other square of the rectangle to form a domino of type (a,b), so the rectangle cannot be (a,b)-tileable.

4. **[30]** Prove that an $m \times n$ rectangle is (b,b)-tileable if and only if $2b \mid m$ and $2b \mid n$.

   Solution: Color the first b rows of an $m \times n$ rectangle black, the next b white, the next b black, etc. Any (b,b) domino covers one square of each color, so for the rectangle to be (b,b)-tileable, there must be the same number of black squares as white squares. This is possible only when $2b \mid m$. Similarly, we must have $2b \mid n$. It is easy to exhibit a tiling of all such rectangles, proving the claim. (It is also possible to prove this using the lemma described below.)

5. **[35]** Prove that an $m \times n$ rectangle is $(0,b)$-tileable if and only if $2b \mid m$ or $2b \mid n$.

   Solution: It is easy to exhibit a tiling of such a rectangle. The other direction follows from below. (It is also possible to prove this using a coloring argument as above: starting in one corner, divide the board into $b \times b$ blocks and color them checkerboard fashion. The details are left to the reader.)

6. **[40]** Let k be an integer such that $k \mid a$ and $k \mid b$. Prove that if an $m \times n$ rectangle is (a,b)-tileable, then $2k \mid m$ or $2k \mid n$.

   Solution: We prove the following lemma.

   Lemma. *Let k be a positive integer such that $k|a$ and $k|b$. Then an $m \times n$ rectangle is (a,b)-tileable if and only if an $m' \times n'$ rectangle is $(\frac{a}{k}, \frac{b}{k})$-tileable for $\lfloor \frac{m}{k} \rfloor \leq m' \leq \lceil \frac{m}{k} \rceil$ and $\lfloor \frac{n}{k} \rfloor \leq n' \leq \lceil \frac{n}{k} \rceil$. (Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, while $\lceil x \rceil$ denotes the least integer greater than or equal to x.)*

   Proof. Number the rows and columns in order. For each pair $0 \leq i, j < k$, consider the set of squares in a row congruent to i modulo k and in a column congruent to j modulo k. If one square of a type (a,b) domino lies in this set, then so does the other. We can therefore partition the rectangle into these sets and then tile these sets instead. Each such set is a rectangular array of dimensions $m' \times n'$, with $\lfloor \frac{m}{k} \rfloor \leq m' \leq \lceil \frac{m}{k} \rceil$ and $\lfloor \frac{n}{k} \rfloor \leq n' \leq \lceil \frac{n}{k} \rceil$, and a type (a,b) domino on the original rectangle is a type $(\frac{a}{k}, \frac{b}{k})$ domino on this new array. Since all possible pairs (m', n') occur, the result follows. □

   Suppose $2k \nmid m$ and $2k \nmid n$. Then at least one of $\lfloor \frac{m}{k} \rfloor$ and $\lceil \frac{m}{k} \rceil$ is odd, so we can choose m' odd. Likewise we can choose n' odd. But then an $m' \times n'$ rectangle has odd area and so cannot be tileable, implying that the $m \times n$ rectangle is not tileable.

An Interlude — Discovering One's Roots [100]

A *kth root of unity* is any complex number ω such that $\omega^k = 1$.

7. **[15]** Find a real, irreducible quartic polynomial with leading coefficient 1 whose roots are all twelfth roots of unity.

   Solution: All twelfth roots of unity are roots of

$$\begin{aligned} x^{12} - 1 &= (x^6 - 1)(x^6 + 1) \\ &= (x^3 - 1)(x^3 + 1)(x^6 + 1) \\ &= (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)(x^2 + 1)(x^4 - x^2 + 1), \end{aligned}$$

   so the answer is $x^4 - x^2 + 1$.

8. **[25]** Let x and y be two kth roots of unity. Prove that $(x + y)^k$ is real.

   Solution: Note that

$$\begin{aligned} (x + y)^k &= \sum_{i=0}^{k} \binom{k}{i} x^i y^{k-i} \\ &= \frac{1}{2} \sum_{i=0}^{k} \binom{k}{i} (x^i y^{k-i} + x^{k-i} y^i) \end{aligned}$$

   by pairing the ith and $(k - i)$th terms. But $x^{k-i} y^i = (x^i y^{k-i})^{-1}$ since x and y are kth roots of unity. Moreover, since x and y have absolute value 1, so does $x^i y^{k-i}$, so $x^{k-i} y^i$ is in fact its complex conjugate. It follows that their sum is real, thus so is $(x + y)^k$.

   This can also be shown geometrically. The argument of x (the angle between the vector x and the positive x-axis) is an integer multiple of $\frac{2\pi}{k}$, as is the argument of y. Since $x + y$ bisects the angle between x and y, its argument is an integer multiple of $\frac{\pi}{k}$. Multiplying this angle by k gives a multiple of π, so $(x + y)^k$ is real.

9. **[30]** Let x and y be two distinct roots of unity. Prove that $x + y$ is also a root of unity if and only if $\frac{y}{x}$ is a cube root of unity.

   Solution: This is easiest to see geometrically. The vectors corresponding to x, y, and $-x - y$ sum to 0, so they form a triangle. In order for them all to be roots of unity, they must all have length one, so the triangle must be equilateral. Therefore the angle between x and y is $\pm\frac{2\pi}{3}$, that is, $\frac{y}{x}$ is a cube root of unity.

10. **[30]** Let x, y, and z be three roots of unity. Prove that $x + y + z$ is also a root of unity if and only if $x + y = 0$, $y + z = 0$, or $z + x = 0$.

   Solution: Again, we consider the geometric picture. Arrange the vectors x, y, z, and $-x - y - z$ so as to form a quadrilateral. If they are all roots of unity, they form a quadrilateral all of whose side lengths are 1. If the quadrilateral is degenerate, then two of the vectors sum to 0, which implies the result. But even if it is not degenerate, the quadrilateral must be a rhombus, and since opposite sides of a rhombus are parallel, this again implies that two of the four roots of unity sum to 0.

Early Re-tile-ment [**150**]

Let $S = \{s_0, \ldots, s_n\}$ be a finite set of integers, and define $S + k = \{s_0 + k, \ldots, s_n + k\}$. We say that S and T are *equivalent*, written $S \sim T$, if $T = S + k$ for some k. Given a (possibly infinite) set of integers A, we say that S *tiles* A if A can be partitioned into subsets equivalent to S. Such a partition is called a *tiling* of A by S.

11. **[20]** Find all sets S with minimum element 1 that tile $A = \{1, \ldots, 12\}$.

Solution: This can be done by brute force. Alternatively, note that if $P(x)$ and $Q(x)$ are polynomials with coefficients either 0 or 1 with $P(x)Q(x) = x + x^2 + \cdots + x^{12}$, then the set consisting of the exponents of nonzero terms in P tiles A. Either way, we find that S is one of the following: $\{1\}$, $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,7\}$, $\{1,2,3\}$, $\{1,3,5\}$, $\{1,5,9\}$, $\{1,2,3,4\}$, $\{1,2,7,8\}$, $\{1,4,7,10\}$, $\{1,2,3,4,5,6\}$, $\{1,2,3,7,8,9\}$, $\{1,2,5,6,9,10\}$, $\{1,3,5,7,9,11\}$, or A itself.

12. **[35]** Let A be a finite set with more than one element. Prove that the number of nonequivalent sets S which tile A is always even.

Solution: Suppose A can be partitioned into sets $S_0, \ldots, S_m$, each equivalent to S. (This partition is unique, simply by choosing S_0 to contain the smallest element of A, S_1 the smallest element of A not in S_0, etc.) Then if $S_j = S + t_j$, each element of A can be written uniquely as $s_i + t_j$ for some i and j. But then the set T containing all t_j also tiles A by translation by the s_i. We cannot have S and T equivalent, for if so, since A has more than one element, both S and T would as well. This would imply that $s_0 + t_1 = s_1 + t_0$, an overlap in the tiling of A. We can thus pair together S and T, each of which tile A, so that the total number of sets tiling A must be even.

13. **[25]** Exhibit a set S which tiles the integers $\mathbf{Z}$ but not the natural numbers $\mathbf{N}$.

Solution: One example is $\{1,3,4,6\}$. Since its elements are all distinct modulo 4, it tiles $\mathbf{Z}$ by translation by multiples of 4. On the other hand, it is easy to see that it cannot tile $\mathbf{N}$: 1 is contained in $\{1,3,4,6\}$, but then there is no possible set for 2 to be contained in that does not overlap. Another example is $\{1,2,6\}$.

14. **[30]** Suppose that S tiles the set of all integer cubes. Prove that S has only one element.

Solution: Let the difference between the smallest and largest element of S be a. Then the set equivalent to S that contains b^3 can only contain integers between $b^3 - a$ and $b^3 + a$, inclusive. But for sufficiently large b, b^3 is the only cube in this range, so S can only have one element.

15. **[40]** Suppose that S tiles the set of odd prime numbers. Prove that S has only one element.

Solution: Consider the set S_0 equivalent to S that contains 3. If it contains 5 but not 7, then the set S_1 equivalent to S containing 7 must contain 9, which is not prime. Likewise, S_0 cannot contain 7 but not 5, because then the set S_1 containing 5 must contain 9. Suppose S_0 contains 3, 5, and 7. Then any other set S_1 of the tiling contains elements p, $p+2$, and $p+4$. But not all of these can be prime, because one of them is divisible by 3. Finally, suppose S_0 contains 3 and has second-smallest element $p > 7$. Then the set S_1 containing 5 does not contain 7 but does contain $p+2$, and the set S_2 containing 7 contains $p+4$. But as before, not all of p, $p+2$, and $p+4$ can be prime. Therefore S has no second-smallest element, so it has only one element.