IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Calculus Test: Solutions

1. A nonzero polynomial $f(x)$ with real coefficients has the property that $f(x) = f'(x)f''(x)$. What is the leading coefficient of $f(x)$?

   Answer: $\frac{1}{18}$
   Solution: Suppose that the leading term of $f(x)$ is cx^n, where $c \neq 0$. Then the leading terms of $f'(x)$ and of $f''(x)$ are cnx^{n-1} and $cn(n-1)x^{n-2}$, respectively, so $cx^n = cnx^{n-1} \cdot cn(n-1)x^{n-2}$, which implies that $n = (n-1)+(n-2)$, or $n = 3$, and $c = cn \cdot cn(n-1) = 18c^2$, or $c = \frac{1}{18}$.

2. Compute $\lim_{x \to 0} \dfrac{e^{x \cos x} - 1 - x}{\sin(x^2)}$.

   Answer: $\frac{1}{2}$
   Solution: Let's compute all the relevant Maclaurin series expansions, up to the quadratic terms:

$$x \cos x = x + \ldots, \qquad e^{x \cos x} = 1 + x + \frac{1}{2}x^2 + \ldots, \qquad \sin(x^2) = x^2 + \ldots,$$

   so

$$\lim_{x \to 0} \frac{e^{x \cos x} - 1 - x}{\sin(x^2)} = \lim_{x \to 0} \frac{\frac{1}{2}x^2 + \ldots}{x^2 + \ldots} = \frac{1}{2}.$$

3. At time 0, an ant is at $(1,0)$ and a spider is at $(-1,0)$. The ant starts walking counterclockwise along the unit circle, and the spider starts creeping to the right along the x-axis. It so happens that the ant's horizontal speed is always half the spider's. What will the shortest distance ever between the ant and the spider be?

   Answer: $\frac{\sqrt{14}}{4}$
   Solution: Picture an instant in time where the ant and spider have x-coordinates a and s, respectively. If $1 \leq s \leq 3$, then $a \leq 0$, and the distance between the bugs is at least 1. If $s > 3$, then, needless to say the distance between the bugs is at least 2. If $-1 \leq s \leq 1$, then $s = 1 - 2a$, and the distance between the bugs is

$$\sqrt{(a-(1-2a))^2 + (1-a^2)} = \sqrt{8a^2 - 6a + 2} = \sqrt{\frac{(8a-3)^2+7}{8}},$$

   which attains the minimum value of $\sqrt{7/8}$ when $a = 3/8$.

4. Compute $\displaystyle\sum_{k=1}^{\infty} \frac{k^4}{k!}$.

   Answer: $15e$
   Solution: Define, for non-negative integers n,

$$S_n := \sum_{k=0}^{\infty} \frac{k^n}{k!},$$

where $0^0 = 1$ when it occurs. Then $S_0 = e$, and, for $n \geq 1$,

$$S_n = \sum_{k=0}^{\infty} \frac{k^n}{k!} = \sum_{k=1}^{\infty} \frac{k^n}{k!} = \sum_{k=0}^{\infty} \frac{(k+1)^n}{(k+1)!} = \sum_{k=0}^{\infty} \frac{(k+1)^{n-1}}{k!} = \sum_{i=0}^{n-1} \binom{n-1}{i} S_i,$$

so we can compute inductively that $S_1 = e$, $S_2 = 2e$, $S_3 = 5e$, and $S_4 = 15e$.

5. Compute

$$\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$$

Answer: $5 - 6 \ln 2$
Solution: Writing $x = u^6$ so that $dx = 6u^5 du$, we have

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}} &= \int_0^1 \frac{6u^5 du}{u^3 + u^2} \\ &= 6 \int_0^1 \frac{u^3 du}{u+1} \\ &= 6 \int_0^1 \left(u^2 - u + 1 - \frac{1}{u+1} \right) du \\ &= 6 \left(\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1| \Big|_0^1 \right) = 5 - 6\ln(2) \end{aligned}$$

6. A triangle with vertices at $(1003, 0)$, $(1004, 3)$, and $(1005, 1)$ in the xy-plane is revolved all the way around the y-axis. Find the volume of the solid thus obtained.

Answer: 5020π
Solution: Let $T \subset \mathbb{R}^2$ denote the triangle, including its interior. Then T's area is $5/2$, and its centroid is $(1004, 4/3)$, so

$$\int_{(x,y)\in T} x \, dx \, dy = \frac{5}{2} \cdot 1004 = 2510.$$

We are interested in the volume

$$\int_{(x,y)\in T} 2\pi x \, dx \, dy,$$

but this is just $2\pi \cdot 2510 = 5020\pi$.

7. Find all positive real numbers c such that the graph of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - cx$ has the property that the circle of curvature at any local extremum is centered at a point on the x-axis.

Answer: $\frac{\sqrt{3}}{2}$
Solution: The equation $0 = f'(x) = 3x^2 - c$ has two real roots: $\pm\sqrt{c/3}$. Let $a := \sqrt{c/3}$. As $f''(-a) = -6\sqrt{c/3} < 0$, f has a unique local maximum at $x = -a$.

Because f has half-turn symmetry about the origin, it suffices to consider this local extremum. The radius of curvature at any local extremum is

$$r(x) = \frac{1}{|f''(x)|} = \frac{1}{6|x|},$$

so the condition in the problem is equivalent to

$$\begin{aligned} r(-a) &= f(-a) \\ \frac{1}{6a} &= -a(a^2 - c) \\ 1 = 6a^2(c - a^2) &= 2c(2c/3) \\ c &= \sqrt{3}/2. \end{aligned}$$

8. Compute

$$\int_0^{\pi/3} x \tan^2(x) dx$$

Answer: $\frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{18} - \ln 2$
Solution: We have

$$\begin{aligned} \int_0^{\pi/3} x \tan^2(x) dx &= \int_0^{\pi/3} x \left(-1 + \frac{1}{\cos^2(x)}\right) dx \\ &= -\frac{x^2}{2}\bigg|_0^{\pi/3} + \int_0^{\pi/3} \frac{x dx}{\cos^2(x)} \\ &= -\frac{x^2}{2}\bigg|_0^{\pi/3} + \left(x\tan(x)\bigg|_0^{\pi/3} - \int_0^{\pi/3} \tan(x) dx \right) \quad (u = x;\ dv = \frac{dx}{\cos^2(x)}) \\ &= -\frac{x^2}{2} + x\tan(x) + \ln|\cos(x)|\ \bigg|_0^{\pi/3} = -\frac{\pi^2}{18} + \frac{\pi\sqrt{3}}{3} - \ln(2) \end{aligned}$$

9. Compute the sum of all real numbers x such that

$$2x^6 - 3x^5 + 3x^4 + x^3 - 3x^2 + 3x - 1 = 0$$

Answer: $-\frac{1}{2}$
Solution: The carefully worded problem statement suggests that repeated roots might be involved (not to be double counted), as well as complex roots (not to be counted). Let $P(x) = 2x^6 - 3x^5 + 3x^4 + x^3 - 3x^2 + 3x - 1$. Now, a is a double root of the polynomial $P(x)$ if and only if $P(a) = P'(a) = 0$. Hence, we consider the system

$$\begin{aligned} P(a) = 2a^6 - 3a^5 + 3a^3 + a^3 - 3a^2 + 3a - 1 &= 0 \\ P'(a) = 12a^5 - 15a^4 + 12a^3 + 3a^2 - 6a + 3 &= 0 \\ \implies 3a^4 + 8a^3 - 15a^2 + 18a - 7 &= 0 \\ 37a^3 - 57a^2 + 57a - 20 &= 0 \\ a^2 - a + 1 &= 0 \end{aligned}$$

We have used polynomial long division to deduce that any double root must be a root of $a^2 - a + 1$! With this information, we can see that $P(x) = (x^2 - x + 1)^2(2x^2 + x - 1)$. The real roots are easily computed via the quadratic formula, leading to an answer of $-\frac{1}{2}$. In fact the repeated roots were complex.

10. Suppose f and g are differentiable functions such that

$$xg(f(x))f'(g(x))g'(x) = f(g(x))g'(f(x))f'(x)$$

for all real x. Moreover, f is nonnegative and g is positive. Furthermore,

$$\int_0^a f(g(x))dx = 1 - \frac{e^{-2a}}{2}$$

for all reals a. Given that $g(f(0)) = 1$, compute the value of $g(f(4))$.

Answer: e^{-16} or $\frac{1}{e^{16}}$
Solution: Differentiating the given integral with respect to a gives $f(g(a)) = e^{-2a}$. Now

$$x\frac{d\left[\ln\left(f(g(x))\right)\right]}{dx} = x\frac{f'(g(x))g'(x)}{f(g(x))} = \frac{g'(f(x))f'(x)}{g(f(x))} = \frac{d\left[\ln\left(g(f(x))\right)\right]}{dx}$$

where the second equals sign follows from the given. Since $\ln\left(f(g(x))\right) = -2x$, we have $-x^2 + C = \ln\left(g(f(x))\right)$, so $g(f(x)) = Ke^{-x^2}$. It follows that $K = 1$ and $g(f(4)) = e^{-16}$.